Variants of Mitchell Forcing

Dissertation zur Erlangung des Doktorgrades

vorgelegt von

Hannes Valentin Jakob

an der Fakultät für Mathematik und Physik
der Universität Freiburg

Freiburg, Februar 2025

Hannes Jakob: Variants of Mitchell Forcing
Dekan der Fakultät: Prof. Dr. Michael Růžička
1. Gutachterin: Prof. Dr. Heike Mildenberger
2. Gutachter: Prof. Dr. John Krueger
Datum der mündlichen Prüfung: 31.01.2025

Abstract of the Dissertation

Variants of Mitchell Forcing

von

Hannes Valentin Jakob

Zur Erlangung des Doktorgrades
Doktor Rerum Naturalium in Mathematik
Universität Freiburg, 2025

This thesis aims to answer questions in the area of infinitary combinatorics through the usage of variants of Mitchell forcing. Mitchell forcing was originally conceived by William Mitchell to construct a model where a successor cardinal has the tree property. Recently, it has been found to also have applications to other problems. By introducing new tools, most notably a general framework for working with arbitrary orders on products of sets and a general variant of Mitchell forcing, we obtain solutions to questions posed by John Krueger and Matthew Foreman regarding variants of internal approachability as well as by Rahman Mohammadpour regarding guessing models and the slender list property.

Für meinen Großvater Karl-Ernst Kiel,
dem ich meine Liebe zur Mathematik zu verdanken habe

Table of Contents

1 Introduction **1**

1.1 Organization of the Thesis 1

1.2 Contributions of the Author 6

1.3 A List of the Variants of Mitchell Forcing 8

2 Preliminaries **9**

2.1 Forcing 9

2.2 Clubs and Stationary Subsets of $[\lambda]^{<\kappa}$ 12

2.3 Thin and Slender Lists 14

2.4 Variants of Internal Approachability 16

3 Strong Distributivity **18**

3.1 Definition and Basic Results 18

3.2 A Game Characterisation of Strong Distributivity 19

3.3 The Strong Easton Lemma and Stationary Preservation 21

3.4 Very Fat Sets 24

4 Orders on Products **29**

4.1 Working with Arbitrary Orders on Products of Sets 29

4.2 The Approximation Property 32

4.3 Examples of Strongly Distributive Forcings 35

5 Embedding Characterisations of Small Large Cardinals **37**

5.1 General Results 37

5.2 Elementary Submodels Witnessing Large Cardinals 39

5.3 Small Models and Forcing 42

5.4 Laver Diamonds for Small Large Cardinals 45

6 A General Mitchell Forcing **50**

6.1 Defining $\mathbb{M}$ 50

6.2 The Properties of $\mathbb{M}$ 51

7 The Internal Structure of Elementary Submodels **57**
7.1 Preliminary Results 59
7.2 Distinguishing Internal Stationarity and Clubness 63
7.2.1 A Single Cardinal 63
7.2.2 Infinitely Many Cardinals 66
7.3 Distinguishing Internal Clubness and Approachability 70
7.3.1 A Single Cardinal 70
7.3.2 Infinitely Many Cardinals 73
7.4 Cascading Variants of Internal Approachability 76

8 On the Ineffable Slender Property **82**
8.1 Two ISP Preservation Theorems 82
8.2 ISP at Weakly Inaccessible Cardinals 85
8.3 Specifying the Slenderness below κ 85
8.4 ISP and Cardinal Arithmetic 87
8.5 Internally Club Guessing Models 89
8.6 Making ISP Indestructible Under Directed-Closed Forcing 92
8.7 ISP and no Disjoint Stationary Sequence 94
8.8 ISP and Club Stationary Reflection 98

9 Conclusion and Open Questions **103**
9.1 Strong Distributivity 103
9.2 Variants of Internal Approachability 103
9.3 The Ineffable Slender Property 105

Acknowledgments

First and foremost, I want to thank my advisor, Prof. Heike Mildenberger. My first ever introduction to mathematical logic was a course given by her which I took in the second semester of my Bachelor's degree (I incidentally attended the lecture because I was hoping it would be an easy introduction to propositional logic). Despite finding the course material very difficult, I became deeply interested in logic and also attended her set theory lecture in the following semester, falling in love with the subject. Since then, she has guided my mathematical journey and advised all of my theses, from the Bachelor's and Master's thesis to this dissertation now. During this time, she has been an excellent mentor and helped me grow by proofreading early drafts of my papers and through fruitful discussions about mathematics.

I also want to thank the external referee, Prof. John Krueger, for his very detailed and immensely helpful referee report, leading to improvement of this thesis.

Next, I want to thank Dr. Maxwell Levine. My first introduction to Mitchell forcing was during a course given by him on the use of large cardinals in set theory. Afterwards, it was a talk he gave during the first semester of my PhD (”On Disjoint Stationary Sequences”) which led me to my first research question, namely whether the principle ISP implied the existence of a disjoint stationary sequence. Working on this question was the start of real mathematical research for me and helped me write my first paper (albeit on a different, but related problem). Afterwards, Maxwell and I started working together and our weekly discussions about mathematics and academia in general gave me a new-found understanding of the world that I intend to work in.

I want to thank my parents for raising me, for their support during all of my education and for never suggesting to me that I should study something “real” instead of mathematics. I also want to thank all of the other “non-mathematical” people (i.e. my friends and my two sisters) simply for being in my life, keeping me grounded while doing this untethered research and allowing me to ramble about my mathematical thoughts despite not understanding much.

Lastly, I want to thank Peter Tchaikovsky and Gustav Mahler. Without their music, I am not sure I would have managed all of the days that I spent proofreading and revising this document.

CHAPTER 1

Introduction

A *tree* on a cardinal κ is a set consisting of functions from ordinals below κ into some set such that for any given $\alpha < \kappa$ there is at least one but $<\kappa$ functions with domain α. It might be natural to think that every tree on κ has cofinal branches (a *cofinal branch* is a function f on κ such that $f \upharpoonright \alpha \in T$ for every $\alpha < \kappa$; in this case we say that κ has the *tree property*), but this is actually not the case: While the first infinite cardinal ω does have the tree property by a result of Kőnig, it was discovered by Aronszajn that there always exists a tree on ω_1 without a cofinal branch (this might seem unintuitive, but consider a tree where all the functions map injectively into the rational numbers; a cofinal branch could be used to construct an injection from ω_1 into $\mathbb{Q}$, a contradiction). Additionally, a similar construction works to construct a κ^+-tree without a cofinal branch whenever $2^{<\kappa} = \kappa$ (this was shown by Specker). This leaves open the question whether ω_2 can have the tree property. By the previous remark we would need a model where $2^\omega \geq \omega_2$. To answer this question, Mitchell constructed a way to turn an inaccessible cardinal κ with the tree property into ω_2 while also adding κ many subsets of ω in such a way that the tree property of κ is preserved. This forcing became known as *Mitchell forcing*. It can be viewed as an iteration of successively adding subsets to ω and collapsing cardinals below κ to ω_1. In limit steps, we take the limit with "mixed support" to preserve the cardinal ω_1: Cohen Forcing preserves ω_1 because it has small antichains (this remains true if we are taking a limit with finite support) and Levy Collapses preserve ω_1 because they are countably closed (this remains true if we are taking a limit with countable support).

The concept of a *mixed support iteration* was formalized by Krueger who applied it to a very different set of problems concerning the variants of internal approachability of elementary submodels. We will continue his work while working with orders which more closely resemble the original Mitchell forcing. This is because such forcing orders come with a projection analysis which is crucial when trying to obtain results for more than one cardinal simultaneously.

1.1 Organization of the Thesis

The thesis is organized as follows: After reviewing known definitions and results in the first chapter, we have the following chapters containing new work:

Strong Distributivity

We say that a forcing is strongly $<\mu$-distributive if every μ-sequence of open dense sets has a thread. This property lies between $<\mu$-distributivity and μ-strategic closure and has much nicer preservation properties compared to $<\mu$-distributivity (e.g. any strongly $<\mu$-distributive poset preserves stationary subsets of μ and the μ-c.c. of forcing notions). In this chapter, we will introduce strong distributivity, show a game characterization and obtain an improvement to the well-known Easton lemma.

Orders on Products

Much difficulty when working with Mitchell forcing comes from the fact that it behaves not quite like an iteration but also not quite like a product forcing. In this chapter we will introduce techniques for working with arbitrary orders on products of sets which is very helpful in order to prove regularity properties for variants of Mitchell forcing. In combination with the previous chapter, we prove a theorem which generalizes a result of Unger (see [Ung15]) by obtaining the same conclusion under weaker hypotheses:

Theorem A. *Let $(\mathbb{P}\times\mathbb{Q},R)$ be an iteration-like ordering and δ a cardinal. Assume the base ordering $(\mathbb{P},b(R))$ is square-δ-c.c. and the term ordering $(\mathbb{P}\times\mathbb{Q},t(R))$ is strongly $<\delta$-distributive. Then $(\mathbb{P}\times\mathbb{Q},R)$ has the $<\delta$-approximation property.*

Here we say that a forcing $\mathbb{R}$ has the $<\delta$-approximation property if there is no $x\in V[\mathbb{R}]\setminus V$ such that $x\cap y\in V$ for all $y\in[V]^{<\delta}\cap V$. This property was implicit in the work of Mitchell regarding the tree property (see [Mit72]) and later studied explicitely by Hamkins (see [Ham99]). In general, obtaining this property is very tedious, so our result provides an important simplification.

Laver Functions for Large Cardinals

In [HLN19], the authors classify large cardinals through the existence of *small embeddings*, embeddings between transitive sets which do not map from the cardinal, but to it. We will obtain analogues of the well-known Laver Diamond for these embeddings in order to later obtain consistency results with (presumably) optimal assumptions. We also reprove well-known folklore results regarding the interaction between small models and forcing.

A General Variant of Mitchell Forcing

In [Kru08a], Krueger defines the notion of a *mixed support iteration*, an iteration where μ-Cohen forcing is used at even ordinals, some $<\mu^+$-closed forcing is used at odd ordinals and limits are taken using $<\mu$-support for even ordinals and μ-support for odd ordinals. Using forcings of this kind, in [Kru09], Krueger obtained models where various variants of internal approachability are distinct for stationarily many substructures of $H(\Theta)$. In this chapter, we modify Krueger's methods as follows:

1. We allow Cohen forcing and closed forcing to be used at arbitrary ordinals, thereby letting us construct guessing variants of Mitchell forcing to obtain indestructibility results.

2. We obtain a projection analysis which says that any such forcing (using τ-Cohen forcing and $<\mu$-closed forcing) is the projection of $\mathrm{Add}(\tau, X) \times \mathbb{T}$ where X is some set and $\mathbb{T}$ is a $<\mu$-closed poset. This is crucial when trying to obtain consistency results for successive cardinals.

Despite these changes, the forcings still have very similar preservation properties to Krueger's mixed support iterations. Most notably, thanks to Theorem A, many quotients of the whole forcing by initial segments have the $<\tau^+$-approximation property.

The Internal Structure of Elementary Submodels

In [FT05], the authors introduced the following variants of internal approachability:

Definition. Let $\mu < \Theta$ be cardinals and $N \in [H(\Theta)]^\mu$.

1. N is *internally unbounded* if $[N]^{<\mu} \cap N$ is unbounded in $[N]^{<\mu}$
2. N is *internally stationary* if $[N]^{<\mu} \cap N$ is stationary in $[N]^{<\mu}$
3. N is *internally club* if $[N]^{<\mu} \cap N$ contains a club in $[N]^{<\mu}$
4. N is *internally approachable* if there exists an increasing and continuous sequence $(N_i)_{i<\mu}$ such that $\bigcup_{i<\mu} N_i = N$ and for every $j < \mu$, $(N_i)_{i<j} \in N$ and $N_j \in [N]^{<\mu}$.

In the same paper, the authors ask if these properties can consistently be distinct. This was answered by Krueger in a series of papers:

1. In [Kru07] Krueger shows PFA implies that for any $\lambda \geq \omega_2$ there exist stationarily many $N \in [H(\lambda)]^{\omega_1}$ that are internally club but not internally approachable.

2. In [Kru08b] he shows that MM implies that for any $\lambda \geq \omega_2$ there exist stationarily many $N \in [H(\lambda)]^{\omega_1}$ that are internally unbounded but not internally stationary and that PFA^{+2} implies that for any $\lambda \geq \omega_2$ there are stationarily many $N \in [H(\lambda)]^{\omega_1}$ that are internally stationary but not internally club.

3. In [Kru08c] he produces, from a supercompact cardinal κ and any regular $\mu < \kappa$, a forcing extension where there are, for any $\lambda \geq \mu^+$, stationarily many $N \in [H(\lambda)]^\mu$ which are internally club but not internally approachable.

4. In [Kru09] he uses the methods he developed in [Kru08a] to produce, from a supercompact cardinal κ and any regular $\mu < \kappa$, a forcing extension where there are, for any $\lambda \geq \mu^+$, stationarily many $N \in [H(\lambda)]^\mu$ which are internally stationary but not internally club.

In [Kru09] he asks if these distinctions can be obtained simultaneously for many successive cardinals and if the assumption of supercompactness can be relaxed. In this chapter (partially joint with Maxwell Levine) we answer these questions in the affirmative. Namely, we show:

Theorem B. *Assume $(\kappa_n)_{n\in\omega}$ is an increasing sequence of Mahlo cardinals. There exist (separate) forcing extensions in which $\kappa_n = \aleph_{n+2}$ and the following holds for every $n \geq 0$:*

1. *For every $\Theta > \aleph_{n+1}$ there are stationarily many $N \in [H(\Theta)]^{\aleph_{n+1}}$ which are internally stationary but not internally club.*
2. *For every $\Theta > \aleph_{n+1}$ there are stationarily many $N \in [H(\Theta)]^{\aleph_{n+1}}$ which are internally club but not internally approachable.*

This provides a drastic weakening of the previously assumed consistency strength. Additionally, there are two interesting differences when comparing our result with a similar result concerning the tree property (see [CF98], where a model is produced in which every $\aleph_{n+2}$, $n \in \omega$ has the tree property): Firstly, we obtain the result without an increase in consistency strength: The tree property at one cardinal requires just a weakly compact cardinal while the tree property at two successive cardinals has much higher consistency strength than that of two weakly compact cardinals. In our case we need one Mahlo cardinal for the result at one cardinal (this is provably optimal) and just infinitely many Mahlo cardinals for the result at infinitely many successive cardinals. This is due to the fact that Mahloness is much easier preserved than weak compactness. Secondly, we can obtain the result using a product of Mitchell forcings which heavily simplifies the presentation: In the case of the tree property, an iteration of Mitchell forcings is used which requires some additional techniques (taking the Cohen forcing from an inner model because the first Mitchel forcing violates necessary cardinal arithmetic assumptions). This is due to the fact that our results are much easier preserved "downwards" when compared to the tree property.

We also investigate the variants of internal approachability from a different viewpoint: Foreman asked in [For05] if a model $M \prec H(\omega_3)$ of size ω_1 must be internally stationary provided that $M \cap H(\omega_2)$ is internally stationary. While our solution to this problem (answering it in the negative) does not fit into the context of this thesis (since it is not obtained using a variant of Mitchell forcing), the question prompted an investigation into related problems, which can be solved using variants of Mitchell forcing. We show:

Theorem C. *Assume κ is κ^+-ineffable and $\mu < \kappa$ is regular. There exist (separate) forcing extensions where $\kappa = \mu^+$ and the following holds:*

1. *There are stationarily many $N \in [H(\mu^{++})]^\mu$ such that $N \cap H(\mu^+)$ is internally approachable but N is not internally approachable.*
2. *$\mu^+ \in I[\mu^+]$ and there are stationarily many $N \in [H(\mu^+)]^\mu$ which are internally stationary but not internally club.*

The second statement of the above theorem contrasts the following result:

Theorem (Folklore). *Assume μ is regular and $2^\mu = \mu^+$ Then $\mu^+ \notin I[\mu^+]$ if and only if there are stationarily many $N \in [H(\mu^+)]^\mu$ which are internally unbounded but not internally approachable.*

A related result by Krueger (see [Kru09, Theorem 6.5]) states that again under $2^\mu = \mu^+$ there is a disjoint stationary sequence on μ^+ if and only if there are stationarily many $N \in [H(\mu^+)]^\mu$ which are internally unbounded but not internally club. Since $\mu^+ \in I[\mu^+]$ implies that there is no disjoint stationary sequence on μ^+, we have shown that Krueger's theorem also relies on the assumption $2^\mu = \mu^+$.

On the Ineffable Slender Property

In [Wei10], Weiß defined the following generalization of a tree on κ:

Definition. Let $\delta \leq \kappa \leq \lambda$ be cardinals and $f\colon [\lambda]^{<\kappa} \to [\lambda]^{<\kappa}$. f is a *δ-slender (κ, λ)-list* if

1. $f(x) \subseteq x$ for any $x \in [\lambda]^{<\kappa}$
2. For any sufficiently large Θ there is a club $C \subseteq [H(\Theta)]^{<\kappa}$ such that whenever $M \in C$ and $x \in [\lambda]^{<\delta} \cap M$, $f(M \cap \lambda) \cap x \in M$.

$\mathrm{ISP}(\delta, \kappa, \lambda)$ states that whenever f is a δ-slender (κ, λ)-list there is $b \subseteq \lambda$ such that

$$S := \{x \in [\lambda]^{<\kappa} \mid f(x) = x \cap b\}$$

is stationary.

The concept of $<\delta$-slenderness is clearly connected to the $<\delta$-approximation property.

Despite its short history, there are many results regarding the *ineffable slender list property* ISP: Weiß in [Wei10] gave a supposed proof that $\mathrm{ISP}(\omega_1, \omega_2, \geq \omega_2)$ (i.e. $\mathrm{ISP}(\omega_1, \omega_2, \lambda)$ for any $\lambda \geq \omega_2$) is consistent from a supercompact cardinal and showed that it actually follows directly from PFA. However, Holy, Lücke and Njegomir in [HLN19] noticed some problems with Weiß' supposed proof and gave a correct argument.

When compared to the tree property (or even its higher variants), ISP has much stronger implications:

1. Weiß showed in [Wei10] that $\mathrm{ISP}(\omega_1, \omega_2, \omega_2)$ implies that the approachability property fails at ω_1, i.e. $\omega_2 \notin I[\omega_2]$ (it was noticed that $\mathrm{ISP}(\omega_2, \omega_2, \omega_2)$ also suffices).
2. Krueger showed in [Kru19a] that $\mathrm{ISP}(\omega_1, \kappa, \geq \kappa)$ implies that the SCH holds above κ, building on results of Viale in [Via12].
3. Lambie-Hanson and Stejskalová showed in [LHS24a] that $\mathrm{ISP}(\omega_1, \omega_2, \geq \omega_2)$ implies that 2^{ω_1} is as small as possible compared to 2^ω, i.e. $2^{\omega_1} = 2^\omega$ if $\mathrm{cf}(2^\omega) \neq \omega_1$ and $2^{\omega_1} = (2^\omega)^+$ otherwise (we note that this is nontrivial as $\mathrm{ISP}(\omega_1, \omega_2, \geq \omega_2)$ is consistent together with any value δ of 2^ω subject to $\delta \geq \omega_2$ and $\mathrm{cf}(\delta) \neq \omega$).

On the consistency side, many results were obtained using *side condition forcing*, forcing with finite $\in$-sequences of countable models, or its variants. Such forcing orders have many quotients with the $<\omega_1$-approximation property by virtue of being *strongly proper* for many countable structures.

1. Cox and Krueger showed in [CK16] that $\mathrm{ISP}(\omega_1, \omega_2, \geq \omega_2)$ is consistent with any value of 2^ω (as long as its cofinality is $> \omega$ and it itself is $\geq \omega_2$) by taking a product of a particular variant of adequate set forcing with $\mathrm{Add}(\omega, \lambda)$.
2. Mohammadpour and Veličkovič showed in [MV21] that the conjunction of $\mathrm{ISP}(\omega_1, \omega_2, \geq \omega_2)$ and $\mathrm{ISP}(\omega_1, \omega_3, \geq \omega_3)$ is consistent by using a side condition forcing with virtual models with particular closure properties.

Using our study of orders on products and Theorem A, we answer multiple questions raised by Mohammadpour in [Moh23] and obtain other interesting results regarding ISP, showing that, despite many provable implications, there are several statements ISP does not decide. The most important results are summarized as follows:

Theorem D. *Let $\tau < \mu < \kappa \leq \lambda = \lambda^{<\kappa}$ be regular cardinals such that $\tau^{<\tau} = \tau$ and κ is λ-ineffable. There exist (separate) forcing extensions satisfying the following (with $\kappa = \mu^+$):*

1. $\mathrm{ISP}(\tau^+, \kappa, \lambda)$ *holds,* $\mathrm{ISP}(\tau, \kappa, \kappa)$ *fails and 2^τ is an arbitrarily large cardinal below λ.*
2. $\mathrm{ISP}(\kappa, \kappa, \lambda)$ *holds,* $\mathrm{ISP}(\mu, \kappa, \kappa)$ *fails and $2^\tau \neq 2^\mu$.*
3. $\mathrm{ISP}(\tau^+, \kappa, \lambda)$ *holds and there does not exist a disjoint stationary sequence on κ.*
4. $\mathrm{ISP}(\tau^+, \kappa, \lambda)$ *holds as well as club stationary reflection at κ.*

If κ is supercompact (i.e. λ-ineffable for all $\lambda \geq \kappa$), there exists a forcing extension such that:

5. $\mathrm{ISP}(\tau^+, \kappa, \geq \kappa)$ *holds and is indestructible under $<\kappa$-directed closed forcing.*

1.2 Contributions of the Author

The material in this thesis mostly comes from four papers which were written during its creation. Of these papers, three are due to only the author while another one is joint work with Maxwell Levine. In this section, we state chapter-by-chapter which results are from the author.

Preliminaries

This chapter only contains well-known old results.

Strong Distributivity

The concept of strong distributivity was isolated by the author for the first time in [Jak23] and used later in [Jak24b]. Any results, unless stated otherwise, are due to him.

Orders on Products

This framework was also introduced by the author in [Jak23] and used in subsequent work. Again, the results are due to him unless stated otherwise.

Embedding Characterizations of Small Large Cardinals

The results in this chapter are mostly comprised of well-known folklore results (especially in the first section) and results of Holy, Lücke and Njegomir (see [HLN19]). The definition of Π^1_1-correctness, which is modeled after weak compactness, is due to the author, but does not appear in any published work. The results about the interaction of small models and forcing are folklore. The section about Laver diamonds for small large cardinals is again material due to the author and unpublished.

A General Mitchell Forcing

This framework is original material due to the author and unpublished. It draws a lot of inspiration from Krueger's idea of a *mixed support iteration* (see [Kru08a]), but the changes we made (mostly allowing to force with a non-$\mathrm{Add}(\tau)$-poset immediately at limits and modeling the poset more strongly after Mitchell forcing) were necessary for later applications.

The Internal Structure of Elementary Submodels

This chapter contains results from three papers. The model in which there is a distinction between internal stationarity and internal clubness on an interval of cardinals is from [Jak23]. The model in which there is a distinction between internal clubness and internal approachability on an interval of cardinals is from [JL25]. That paper evolved as genuine joint work with all of the results and their proofs stemming from discussions between the author and Maxwell Levine. The approach of using a product of Mitchell forcing is slightly different from the approach in the paper (where an iteration is used instead) and due to the author. The last model in which the approachability property holds despite there being a distinction between internal stationarity and clubness is from [Jak24a].

On the Ineffable Slender Property

This chapter contains results from the paper [Jak24b] and unpublished work due to the author. The first theorem is adapted from Holy, Lücke and Njegomir (see [HLN19]) with a similar proof while the second theorem is a generalization of the first one. The following applications of the first two theorems are due to the author and partially unpublished.

1.3 A List of the Variants of Mitchell Forcing

As the title suggests, this thesis aims to build variants of Mitchell forcing. In this section we will list those variants and their basic ideas.

- $\mathbb{M}_0$ See Definition 7.2.1. This is pretty much the original version used by Mitchell in his consistency proof of the tree property at ω_2 (see [Mit72]).
- $\mathbb{M}_1$ See Definition 7.3.1. This poset is similar to $\mathbb{M}_0$ but uses a different collapsing forcing to ensure the ground model remains club in $[\delta]^{<\mu}$ after δ is collapsed (this is necessary because we are first adding a Cohen subset to ensure non-internal approachability).
- $\mathbb{M}_2$ See Definition 7.4.3. This poset incorporates both collapses used by $\mathbb{M}_0$ and $\mathbb{M}_1$ respectively to obtain models which are internally approachable of different variants at different levels.
- $\mathbb{M}_3$ See Definition 7.4.5. This poset uses the Levy collapse at very specific points to produce models which are not internally club as a whole but have an initial segment which is even internally approachable.
- $\mathbb{M}_4$ See Definition 8.3.1. This is very similar to $\mathbb{M}_0$ but allows larger Cohen forcings because they play a different part in this application (more specifically, in $\mathbb{M}_0$ we needed $\mathrm{Add}(\omega)$ because it forces the ground model to be costationary while in $\mathbb{M}_4$ we only need $\mathrm{Add}(\tau)$ because we want to obtain the $<\tau^+$-approximation property).
- $\mathbb{M}_5$ See Definition 8.4.1. This is simply a product of $\mathbb{M}_4$ and some Cohen forcing and used to show that ISP does not bound the size of the continuum.
- $\mathbb{M}_6$ See Definition 8.4.5. Just like $\mathbb{M}_5$, this is just a product of $\mathbb{M}_4$ and some Cohen forcing and used to show that $\mathrm{ISP}(\tau^{++}, \tau^{++}, \lambda)$ does not imply $2^\tau = 2^{\tau^+}$ even if $\mathrm{cf}(2^\tau) \neq \tau^+$.
- $\mathbb{M}_7$ See Definition 8.5.5. This forcing uses the same collapse as $\mathbb{M}_1$ but collapses the cardinal given by a Laver function l using a club-shooting forcing. Due to this, after forcing with $\mathbb{M}_7$, we have stationarily many guessing models which are internally club.
- $\mathbb{M}_8$ See Definition 8.6.1. This is a guessing variant of $\mathbb{M}_4$ which guesses directed-closed forcings and is used to show that we can make ISP at small cardinals indestructible under directed-closed forcing.
- $\mathbb{M}_9$ See Definition 8.7.5. This is again a guessing variant of $\mathbb{M}_4$. The difference to $\mathbb{M}_8$ is that we guess merely closed forcing and also use a different collapse. Due to this, it forces ISP to be indestructible under the canonical poset forcing $\neg\,\mathrm{DSS}$ and thus shows that ISP does not imply DSS.
- $\mathbb{M}_{10}$ See Definition 8.8.5. This is almost the same as $\mathbb{M}_9$, but using the Levy collapse. We use this forcing to show that ISP is consistent together with club stationary reflection.

CHAPTER 2

Preliminaries

In this chapter, we introduce basic definitions and results that will be used throughout the thesis. Proofs, if omitted, can be found in the standard textbooks on set theory (see [Kun11]; [Jec03]).

2.1 Forcing

We assume the reader is familiar with the basics of forcing. More specialized results regarding forcing will be introduced in this subsection. Our notation is standard, we require filters to be closed upwards (so if $p \leq q$, p forces more).

We start with well-known preservation properties:

Definition 2.1.1. Let $\mathbb{P}$ be a forcing order and κ a cardinal.

1. $\mathbb{P}$ has the *κ-chain condition* (*κ-c.c.*) if $\mathbb{P}$ does not have an antichain of size κ.
2. $\mathbb{P}$ has the *square-κ-c.c.* if and only if $\mathbb{P} \times \mathbb{P}$ (with the product ordering) has the κ-c.c.
3. $\mathbb{P}$ is *κ-Knaster* if whenever $A \subseteq \mathbb{P}$ has size κ, there is $B \subseteq A$ of size κ such that any two elements of B are compatible.
4. $\mathbb{P}$ is *$<\kappa$-centered* if we can write $\mathbb{P} = \bigcup_{\alpha<\mu} \mathbb{P}_\alpha$ such that $\mu < \kappa$ and each $\mathbb{P}_\alpha$ consists of pairwise compatible conditions.

It is a classical result that

$$\mathbb{P} \text{ is } {<}\kappa\text{-centered} \to \mathbb{P} \text{ is } \kappa\text{-Knaster} \to \mathbb{P} \text{ has the square-}\kappa\text{-c.c.} \to \mathbb{P} \text{ has the } \kappa\text{-c.c.}$$

If $\mathbb{P}$ is κ-c.c., then $\mathbb{P}$ does not add a surjection from any ordinal $<\kappa$ to κ. Therefore forcing with a κ-c.c. partial order $\mathbb{P}$ does not collapse cardinals above (and including) κ. A different type of preservation (in this case for small cardinals) comes from certain closure and distributivity assumptions. We first have to define a game played on a partial order:

Definition 2.1.2. Let $\mathbb{P}$ be a partial order and γ an ordinal. The *completeness game on $\mathbb{P}$ of length γ*, denoted $G(\mathbb{P}, \gamma)$, has players INC and COM alternately playing elements of $\mathbb{P}$. The rules are as follows: COM plays at even ordinals (including limits) and has to start by playing $1_{\mathbb{P}}$. After $(p_\alpha)_{\alpha<\gamma}$ has been played, the player whose turn it is has to play an element that is a lower bound of $(p_\alpha)_{\alpha<\gamma}$. COM wins if they can move at every turn $<\gamma$, otherwise INC wins.

If COM has a winning strategy in $G(\mathbb{P}, \kappa+1)$, they have one in $G(\mathbb{P}, \delta)$ for every $\delta < \kappa^+$. However, this is not enough to deduce that they have a winning strategy in $G(\mathbb{P}, \kappa^+)$.

Definition 2.1.3. Let $\mathbb{P}$ be a forcing order and κ a cardinal.

1. $\mathbb{P}$ is $<\kappa$*-distributive* if for every sequence $(D_\alpha)_{\alpha<\mu}$ $(\mu < \kappa)$ of open dense subsets of $\mathbb{P}$, the intersection $\bigcap_{\alpha<\mu} D_\alpha$ is dense.
2. $\mathbb{P}$ is $<\kappa$*-strategically closed* if COM has a winning strategy in $G(\mathbb{P}, \mu)$ for every ordinal $\mu < \kappa$.
3. $\mathbb{P}$ is κ*-strategically closed* if COM has a winning strategy in $G(\mathbb{P}, \kappa)$.
4. $\mathbb{P}$ is $<\kappa$*-closed* if for every descending sequence $(p_\alpha)_{\alpha<\mu}$ $(\mu < \kappa)$ in $\mathbb{P}$, there exists a lower bound.
5. $\mathbb{P}$ is $<\kappa$*-directed closed* if for every directed set $C \subseteq \mathbb{P}$ of size $< \kappa$ (i.e. any two elements of C have a common lower bound in C) there is $q \in \mathbb{P}$ such that $q \leq p$ for every $p \in C$.

As before, the statements are ordered from top to bottom in ascending strength.

The main reason for introducing distributivity is the following:

Lemma 2.1.4. *If $\mathbb{P}$ is $<\kappa$-distributive, G is $\mathbb{P}$-generic and $V[G] \ni f\colon \gamma \to V$ for some $\gamma < \kappa$, $f \in V$.*

So $<\kappa$-distributive forcings cannot collapse cardinals below (and including) κ.

We will also need the following two properties:

Definition 2.1.5. Let $V \subseteq W$ be models of set theory and κ a cardinal.

1. (V, W) has the $<\kappa$*-covering property* if for every set $x \in [V]^{<\kappa} \cap W$, there is $y \in [V]^{<\kappa} \cap V$ such that $x \subseteq y$.
2. (V, W) has the $<\kappa$*-approximation property* if for every set $x \in W$, if $x \cap y \in V$ for every $y \in [V]^{<\kappa} \cap V$, $x \in V$.

If $\mathbb{P}$ is a forcing order, we say that $\mathbb{P}$ has the $<\kappa$-covering (-approximation) property if $(V, V[G])$ has it for every $\mathbb{P}$-generic filter G.

The $<\kappa$-approximation property was first explicitly introduced and studied by Hamkins in [Ham99], but also occurred implicitly in the proof of Mitchell showing the consistency of the tree property at ω_2 (cf. [Mit72], Lemma 3.8).

We need two combinatorial Lemmas that we will use to show that certain forcings do not do too much “damage to the universe“.

Lemma 2.1.6 (Δ-System-Lemma)**.** *Assume that κ is a regular cardinal and $\tau < \kappa$ is such that $\alpha^{<\tau} < \kappa$ for all $\alpha < \kappa$. Let $\mathcal{F}$ be a κ-sized family of sets of cardinality $< \tau$. Then there is $\mathcal{F}' \subseteq \mathcal{F}$ of size κ and r such that $\forall x, y \in \mathcal{F}'$, if $x \neq y$, then $x \cap y = r$.*

Recall the following: $x \subseteq \kappa$ is an *Easton subset of* κ if for all regular $\delta \leq \kappa$, $|x \cap \delta| < \delta$. A cardinal κ is a *Mahlo cardinal* if the set of regular cardinals is stationary in κ.

Lemma 2.1.7 (Easton-Δ-System-Lemma)**.** *Assume that κ is a Mahlo cardinal and $\mathcal{F}$ is a κ-sized family of Easton subsets of κ. Then there is $\mathcal{F}' \subseteq \mathcal{F}$ of size κ and r such that $\forall x, y \in \mathcal{F}'$, if $x \neq y$, then $x \cap y = r$.*

Most of the forcing orders present in this thesis will be built from just three components: The Cohen Forcing (of which we define a specific variant), the Levy Collapse, and a specific club-shooting forcing.

Definition 2.1.8. Let τ be a cardinal and X a set. $\mathrm{Add}(\tau, X)$ consists of $<\tau$-sized partial functions p on X such that for every $x \in \mathrm{dom}(p)$, $p(x) \in \mathrm{Add}(\tau)$, i.e. $p(x)$ is a $<\tau$-sized partial function from τ to 2. Given $p, q \in \mathrm{Add}(\tau, X)$, we let $p \leq q$ if $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and for any $x \in \mathrm{dom}(q)$, $p(x) \supseteq q(x)$.

$\mathrm{Add}(\tau, X)$ is $<\tau$-directed closed and $(2^{<\tau})^+$-Knaster, so if $2^{<\tau} = \tau$, it does not collapse any cardinals. It adds a family $\{a_x \mid x \in X\}$ of distinct subsets of τ.

Definition 2.1.9. Let κ and λ be cardinals. $\mathrm{Coll}(\kappa, \lambda)$ consists of $<\kappa$-sized partial functions p from κ into λ.

If κ is regular, $\mathrm{Coll}(\kappa, \lambda)$ is $<\kappa$-directed closed and $(\lambda^{<\kappa})^+$-Knaster. It adds a surjection from κ to λ. We will also consider this poset for $\lambda < \kappa$ (to simplify later definitions). In this case, it is simply equivalent to $\mathrm{Add}(\kappa, 1)$.

Definition 2.1.10. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing orders. A function $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a *projection* if the following hold:

1. $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$.
2. For all $p' \leq p$, $\pi(p') \leq \pi(p)$
3. For all $p \in \mathbb{P}$, if $q \leq \pi(p)$, there is some $p' \leq p$ such that $\pi(p') \leq q$.

A projection π is trivial if for all $p, p' \in \mathbb{P}$, if $\pi(p) = \pi(p')$, then p and p' are compatible.

If there exists a projection from $\mathbb{P}$ to $\mathbb{Q}$, any extension by $\mathbb{Q}$ can be forcing extended to an extension by $\mathbb{P}$.

Definition 2.1.11. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing orders, $\pi\colon \mathbb{P} \to \mathbb{Q}$ a projection. Let H be $\mathbb{Q}$-generic. In $V[H]$, the forcing order $\mathbb{P}/H$ consists of all $p \in \mathbb{P}$ such that $\pi(p) \in H$. We let $\mathbb{P}/\mathbb{Q}$ be a $\mathbb{Q}$-name for $\mathbb{P}/\dot{H}$ and call $\mathbb{P}/\mathbb{Q}$ the *quotient forcing of* $\mathbb{P}$ *and* $\mathbb{Q}$.

Fact 2.1.12. *Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing orders and $\pi\colon \mathbb{P} \to \mathbb{Q}$ a projection. If H is $\mathbb{Q}$-generic over V and G is $\mathbb{P}/H$-generic over $V[H]$, then G is $\mathbb{P}$-generic over V and $H \subseteq \pi[G]$. In particular, $V[H][G] = V[G]$.*

One checks that if $\pi : \mathbb{P} \to \mathbb{Q}$ is trivial, then $\mathbb{P}/\mathbb{Q}$ is forced to be centered and thus:

Fact 2.1.13. *If there exists a trivial projection $\pi : \mathbb{P} \to \mathbb{Q}$, $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent.*

Lastly, we note that projections behave nicely in combination with products:

Lemma 2.1.14. *Let $\mathbb{P}, \mathbb{Q}$ and $\mathbb{R}$ be forcing orders. Suppose $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a projection. Then there exists a projection $\pi'\colon \mathbb{P} \times \mathbb{R} \to \mathbb{Q} \times \mathbb{R}$ and moreover whenever $G \times H$ is $\mathbb{P} \times \mathbb{R}$-generic, $(\mathbb{P} \times \mathbb{R})/(G \times H)$ is forcing equivalent to $\mathbb{P}/G$.*

Proof. Let $\pi'((p, r)) := (\pi(p), r)$. We verify that π' is a projection. Clearly

$$\pi'(1_{\mathbb{P}\times\mathbb{R}}) = \pi'((1_{\mathbb{P}}, 1_{\mathbb{R}})) = (1_{\mathbb{Q}}, 1_{\mathbb{R}}) = 1_{\mathbb{Q}\times\mathbb{R}}$$

Also π' preserves $\leq$ in both coordinates and thus it preserves $\leq_{\mathbb{P}\times\mathbb{R}}$. Lastly, suppose $(q_1, r_1) \leq \pi'((p_0, r_0)) = (\pi(p_0), r_0)$. Since π is a projection, there is $p_2 \leq p_0$ such that $\pi(p_2) \leq q_1$. It follows that $(p_2, r_1) \leq (p_0, r_0)$ and $\pi'((p_2, r_1)) = (\pi(p_2), r_1) \leq (q_1, r_1)$.

Now assume $G \times H$ is $\mathbb{P} \times \mathbb{R}$-generic. Then by the definition $(\mathbb{P} \times \mathbb{R})/(G \times H)$ consists of all $(p, r) \in \mathbb{P} \times \mathbb{R}$ such that $\pi'((p, r)) = (\pi(p), r) \in G \times H$. Ergo it consists of those pairs (p, r) where $(\pi(p), r) \in G \times H$ which is equal to $(\mathbb{P}/G) \times H$. As H is centered, the map $(p, r) \mapsto p$ is a trivial projection from $(\mathbb{P}/G) \times H$ to $\mathbb{P}/G$. $\square$

2.2 Clubs and Stationary Subsets of $[\lambda]^{<\kappa}$

In his PhD thesis, Christoph Weiss introduced an object called a (κ, λ)-list. To make sense of that definition, we must first introduce Jech's *generalized clubs in the space* $[\lambda]^{<\kappa}$.

Definition 2.2.1. Let κ be a cardinal and X a set of size $\geq \kappa$. $[X]^{<\kappa}$ is defined as the set of all $x \subseteq X$ with size $< \kappa$.

1. $C \subseteq [X]^{<\kappa}$ is *club* in $[X]^{<\kappa}$ if it is unbounded and closed under ascending unions of length $< \kappa$.
2. $S \subseteq [X]^{<\kappa}$ is *stationary* in $[X]^{<\kappa}$ if $S \cap C \neq \emptyset$ for every C which is club in $[X]^{<\kappa}$.

We also define $[X]^{\kappa}$ as the set of all $x \subseteq X$ of size κ. We note that $[X]^{\kappa}$ is club in $[X]^{<\kappa^+}$.

The space $[X]^{<\kappa}$ behaves much like κ: Namely, the intersection of fewer than $\mathrm{cf}(\kappa)$ club sets is club and the club filter is even closed under diagonal intersections, i.e. if C_α is club for every $\alpha \in X$, so is

$$\Delta_{\alpha\in X} C_\alpha := \{x \in [X]^{<\kappa} \mid x \neq \emptyset \wedge x \in \bigcap_{\alpha\in x} C_\alpha\}$$

There is also a connection between clubs in $[X]^{<\kappa}$ and functions $f\colon [X]^{<\omega} \to [X]^{<\kappa}$. If $f\colon [X]^{<\omega} \to [X]^{<\kappa}$, the set

$$C_f := \{x \in [X]^{<\kappa} \mid \forall y \in [x]^{<\omega}\ f(y) \subseteq x\}$$

is club in $[X]^{<\kappa}$. It turns out that these sets generate the club filter (see [Men75]):

Lemma 2.2.2. *Let C be club in $[X]^{<\kappa}$. Then there is $f\colon [X]^{<\omega} \to [X]^{<\kappa}$ such that $C_f \subseteq C$.*

Proof. We define f by induction on $|x|$. If $|x| = 1$, let $f(x)$ be any element of C with $x \subseteq f(x)$. Assume f has been defined for sets of size $< n$ and $|x| = n$. Let $f(x)$ be an element of C containing $f(y)$ for all $y \subseteq x$ of size $< |x|$.

Now assume x is closed under f. It follows that $x = \bigcup A$, where $A := \{f(y) \mid y \in [x]^{<\omega}\}$ Furthermore, A is directed and a subset of C. It follows from [MKS64] that clubs in $[X]^{<\kappa}$ are closed under directed unions. Ergo $x \in C$. $\square$

As a corollary, we obtain that clubs in $[X]^{<\kappa}$ project to sets containing clubs in $[Y]^{<\kappa}$ whenever $Y \subseteq X$:

Corollary 2.2.3. *Let $C \subseteq [X]^{<\kappa}$ be club and $Y \subseteq X$ of size $\geq \kappa$. Then*

$$C \restriction Y := \{N \cap Y \mid N \in C\}$$

contains a club in $[Y]^{<\kappa}$.

Proof. Let $f\colon [X]^{<\omega} \to [X]^{<\kappa}$ be such that $C_f \subseteq C$. Let $g\colon [Y]^{<\omega} \to [Y]^{<\kappa}$ be such that $g(x)$ is the closure of x under f, intersected with Y. Let $N \in [Y]^{<\kappa}$ be closed under g and let $N' \in [X]^{<\kappa}$ be the closure of N under f. Then $N' \cap Y = N$ and $N' \in C_f \subseteq C$. So $C_g \subseteq C \restriction Y$ and thus C_g witnesses that $C \restriction Y$ contains a club. $\square$

Thus if $S \subseteq [Y]^{<\kappa}$ is stationary, $\{N \in [X]^{<\kappa} \mid N \cap Y \in S\}$ is stationary in $[X]^{<\kappa}$. We can also go upwards: If $C \subseteq [Y]^{<\kappa}$, then clearly $\{N \in [X]^{<\kappa} \mid N \cap Y \in C\}$ is club in $[X]^{<\kappa}$. So if $S \subseteq [X]^{<\kappa}$ is stationary, so is $\{N \cap Y \mid N \in S\}$.

We note that forcing with a κ-c.c. forcing orders preserves stationary subsets of $[X]^{<\kappa}$ in a strong way:

Lemma 2.2.4. *Assume $\mathbb{P}$ is κ-c.c. and $\dot{C}$ is a $\mathbb{P}$-name for a club in $[X]^{<\kappa}$. Then there exists D such that D is club in $[X]^{<\kappa}$ and $\Vdash \check{D} \subseteq \dot{C}$.*

Proof. Define

$$D := \{x \in [X]^{<\kappa} \mid \ \Vdash \check{x} \in \dot{C}\}$$

we leave the details to the reader. $\square$

Now we can define our third forcing poset:

Definition 2.2.5. Let $S \subseteq [X]^{<\kappa}$. $\mathbb{P}(S)$ consists of $< \kappa$-sized partial functions p from κ into S such that the following holds:

1. $\mathrm{dom}(p)$ is a successor ordinal.
2. p is $\subseteq$-increasing and continuous.

For $p, q \in \mathbb{P}(S)$, we let $p \leq q$ if $p \supseteq q$.

If S is cofinal in $[X]^{<\kappa}$, $\mathbb{P}(S)$ adds a function $f\colon \kappa \to S$ which is increasing, cofinal and continuous, thus collapsing $|X|$ to κ. Under sufficient conditions (which we will not discuss here), $\mathbb{P}(S)$ is $<\kappa$-distributive and even can preserve stationary subsets of κ.

2.3 Thin and Slender Lists

Definition 2.3.1. Let κ be a cardinal. A *κ-tree* is a set $T \subseteq 2^{<\kappa}$ closed under restriction such that for every $\alpha < \kappa$,

$$1 \leq |\{f \in T \mid \operatorname{dom}(f) = \alpha\}| < \kappa$$

κ has the tree property, written $\mathrm{TP}(\kappa)$, if for every κ-tree T there exists $f \in 2^{\kappa}$ such that $f \upharpoonright \alpha \in T$ for every $\alpha < \kappa$.

The famous Kőnig's lemma implies that every ω-tree has a cofinal branch. However, Aronszajn constructed a tree with countable levels, height ω_1 and no cofinal branch, ergo ω_1 never has the tree property. Aronszajn's result was later generalised by Specker to show that if $2^{<\delta} = \delta$, there is a δ^+-tree without a cofinal branch.

In his PhD thesis (see [Wei10]), Christoph Weiß introduced the notion of a (κ, λ)-list which is based on the idea of a κ-tree:

Definition 2.3.2. Let $\delta \leq \kappa \leq \lambda$ be cardinals. A *(κ, λ)-list* is a function $f\colon [\lambda]^{<\kappa} \to [\lambda]^{<\kappa}$ such that $f(x) \subseteq x$ for every $x \in [\lambda]^{<\kappa}$. We define two notions of thinness:

1. f is *thin* if for every $x \in [\lambda]^{<\kappa}$,
$$|\{f(y) \cap x \mid x \subseteq y \in [\lambda]^{<\kappa}\}| < \kappa$$
2. f is *$<\delta$-slender* if for every large enough Θ, there exists a club $C \subseteq [H(\Theta)]^{<\kappa}$ such that whenever $M \in C$ and $x \in [\lambda]^{<\delta} \cap M$, $f(M \cap \lambda) \cap x \in M$.

We also define two notions of branches of a list f:

1. $b \subseteq \lambda$ is a *cofinal branch* if for all $a \in [\lambda]^{<\kappa}$ there is $z_a \in [\lambda]^{<\kappa}$ such that $a \subseteq z_a$ and $b \cap a = f(z_a) \cap a$.
2. $b \subseteq \lambda$ is an *ineffable branch* if the set
$$\{x \in [\lambda]^{<\kappa} \mid f(x) = b \cap x\}$$
is stationary.

We obtain six different properties:

1. κ has the *λ-super tree property* if every (κ, λ)-list has a cofinal branch.

2. κ has the λ-*ineffable tree property* if every (κ, λ)-list has an ineffable branch.

3. κ has the λ-*super* $<\delta$-*slender property* if every $<\delta$-slender (κ, λ)-list has a cofinal branch.

4. κ has the λ-*ineffable* $<\delta$-*slender property* if every $<\delta$-slender (κ, λ)-list has an ineffable branch.

5. κ has the λ-*super thin property* if every thin (κ, λ)-list has a cofinal branch.

6. κ has the λ-*ineffable thin property* if every thin (κ, λ)-list has an ineffable branch.

In general, at non-inaccessible cardinals κ, the difference between ineffable and cofinal branches comes down purely to the cardinal we chose to collapse. However, the difference between slenderness and thinness depends on the forcing we chose to collapse with. In this thesis, we will only concern ourselves with the λ-ineffable $<\delta$-slender property and write $\mathrm{ISP}(\delta, \kappa, \lambda)$ if κ has the λ-ineffable $<\delta$-slender property.

Every thin list is slender and if κ is inaccessible, every (κ, λ)-list is thin. We will also work with (κ, X)-lists (taking the obvious generalization), noting that every (κ, X)-list is equivalent to a $(\kappa, |X|)$-list.

Lists are connected to seemingly unrelated large cardinal axioms (which we will have to define first).

Definition 2.3.3. Let $\kappa \leq \lambda$ be cardinals.

1. κ is *weakly compact* if for every set algebra B on κ and every κ-complete filter F on B, F can be extended to a κ-complete ultrafilter on B.

2. κ is *strongly compact* if for every $\lambda \geq \kappa$ and every κ-complete filter F on λ, there exists a κ-complete ultrafilter on λ containing F.

3. κ is λ-*supercompact* if there exists an elementary embedding $j\colon V \to M$ with $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$. κ is *supercompact* if it is λ-supercompact for all $\lambda \geq \kappa$.

Theorem 2.3.4. *Let κ be a cardinal.*

1. *(Erdös-Hajnal) κ is weakly compact if and only if κ is inaccessible and has the tree property.*

2. *(Di Prisco-Zwicker [DPZ80], Donder-Weiß [Wei10]) κ is strongly compact if and only if κ is inaccessible and has the super tree property.*

3. *(Donder-Weiß [Wei10], Jech [Jec73] and Magidor [Mag74]) κ is supercompact if and only if κ is inaccessible and has the ineffable tree property.*

Lists also allow us to define the following large cardinal notion:

Definition 2.3.5. Let $\kappa \leq \lambda$ be cardinals. κ is λ-ineffable if κ is inaccessible and has the λ-ineffable tree property.

In particular, a cardinal κ is supercompact if and only if it is λ-ineffable for all $\lambda \geq \kappa$. However, λ-ineffability and λ-supercompactness are not equivalent "level-by-level": An easy calculation shows that any λ-supercompact carries below it a stationary set of λ-ineffable cardinals.

2.4 Variants of Internal Approachability

The approachability ideal was introduced by Shelah in [She79].

Definition 2.4.1. Let κ be a cardinal.

1. If $\overline{a} = (a_\alpha)_{\alpha<\kappa}$ is a sequence of bounded subsets of κ, we say that an ordinal $\gamma < \lambda$ is *approachable with respect to* $\overline{a}$ if there is an unbounded subset $A \subseteq \gamma$ such that
$$\{A \cap \beta \mid \beta < \gamma\} \subseteq \{a_\beta \mid \beta < \gamma\}$$
2. A set $S \subseteq \kappa$ is in $I[\kappa]$ if there is a sequence $\overline{a}$ of bounded subsets of κ and a club C such that every $\gamma \in S \cap C$ is approachable with respect to $\overline{a}$. $I[\kappa]$ is called the *approachability ideal*
3. The *approachability property holds at* κ, written AP_κ, if $I[\kappa^+]$ is improper, i.e. $\kappa^+ \in I[\kappa^+]$.

Related to the approachability ideal is the idea of internal approachability, first introduced and studied by Foreman and Todorcevic in [FT05].

Definition 2.4.2. Let μ be a regular cardinal and N a set of size μ.

1. N is *internally unbounded* if $[N]^{<\mu} \cap N$ is unbounded in $[N]^{<\mu}$,
2. N is *internally stationary* if $[N]^{<\mu} \cap N$ is stationary in $[N]^{<\mu}$,
3. N is *internally club* if $[N]^{<\mu} \cap N$ contains a club in $[N]^{<\mu}$,
4. N is *internally approachable* if there exists an increasing and continuous sequence $(N_i)_{i<\mu}$ of elements of $[N]^{<\mu}$ such that $\bigcup_{i<\mu} N_i = N$ and for every $j < \mu$, $(N_i)_{i<j} \in N$.

Remark 2.4.3. N is internally club if and only if we can write N as an increasing and continuous union of sets $(N_i)_{i<\mu}$ where each N_i is in $[N]^{<\mu} \cap N$.

We will also need a different version of internal approachability:

Definition 2.4.4. Let τ, κ be cardinals, X a set. $N \in [H(\Theta)]^{<\kappa}$ is *internally approachable of length* τ, written $N \in \mathrm{IA}(\tau)$ if there is a continuous chain $(N_i)_{i<\tau}$ of elementary submodels of $H(\Theta)$ such that $N = \bigcup_{i<\tau} N_i$ and for every $j < \tau$, $(N_i)_{i<j} \in N$.

This version differs from the first one by allowing larger N (thus enabling the use of smaller τ). Even though we will later construct non-internally approachable models most of them will be internally approachable in the weaker sense.

We introduced $\mathrm{IA}(\tau)$ because these are the stationary sets which are preserved by posets with sufficient closure:

Lemma 2.4.5. *Let $\tau < \mu \leq \Theta$ be cardinals. Let $S \subseteq [H(\Theta)]^{<\mu} \cap \mathrm{IA}(\tau)$ be stationary and $\mathbb{P}$ a μ-strategically closed poset. Then forcing with $\mathbb{P}$ preserves the stationarity of S.*

A small modification of the proof shows the following:

Lemma 2.4.6. *Let X be a set and $\mu \geq \omega_1$ a regular cardinal. Let $\mathbb{P}$ be a $<\omega_1$-strategically closed and $<\mu$-distributive poset. After forcing with $\mathbb{P}$, $([X]^{<\mu})^V$ is stationary in $[X]^{<\mu}$.*

Proof. Let $\dot{F}$ be a $\mathbb{P}$-name for a function from $[X]^{<\omega}$ into $[X]^{<\mu}$. In V, let Θ be large enough and $(N_i)_{i<\omega}$ a sequence of elementary submodels of $H(\Theta)$ such that $N_i \in N_{i+1}$ and N_0 contains everything relevant. Let $N := \bigcup_{i<\omega} N_i$. In N, define inductively $(p_i)_{i\in\omega}$ where $p_0 := 1_{\mathbb{P}}$, p_{2i+1} decides the closure of $\check{N}_i \cap \check{X}$ under $\dot{F}$ to be some $\check{X}_i$ and p_{2i+2} is played according to a winning strategy in $G(\mathbb{P}, \omega+1)$. By assumption, the sequence $(p_i)_{i\in\omega}$ has a lower bound p. p then forces $\bigcup_{i<\omega} \check{X}_i$ (which is forced to be equal to $\check{N} \cap \check{X}$) to be closed under $\dot{F}$. By assumption, $N \cap X \in ([X]^{<\mu})^V$. □

Again, similarly to the Lemma 2.4.5, one can show:

Lemma 2.4.7. *Let $\mu \leq \kappa$ be regular cardinals. Let $S \subseteq \kappa^+ \cap \mathrm{cof}(<\mu)$ be stationary and $\mathbb{P}$ a $<\mu$-closed poset. Then $\mathbb{P}$ preserves the stationarity of S.*

This is in light of the fact that whenever κ is regular, $\kappa^+ \cap \mathrm{cof}(<\kappa) \in I[\kappa^+]$ (see [She91, Lemma 4.4]).

To obtain results on the internal structure of elementary submodels of $H(\Theta)$, Krueger introduced the following combinatorial objects called disjoint stationary and club sequences.

Definition 2.4.8. Let μ be a cardinal. A sequence $(A_\alpha)_{\alpha\in S}$ is a *disjoint stationary (club)* sequence on μ^+ if the following holds:

1. S is a stationary subset of $\mu^+ \cap \mathrm{cof}(\mu)$.
2. For all $\alpha \in S$, A_α is a stationary (club) subset of $[\alpha]^{<\mu}$
3. For all $\alpha \neq \beta \in S$, $A_\alpha \cap A_\beta = \emptyset$.

Krueger showed that the existence of a disjoint stationary sequence is related to the distinction of internal stationarity and clubness (and the existence of a disjoint club sequence is related to the distinction of internal unboundedness and stationarity). In Chapter 7, we will give a proof of the first fact.

CHAPTER 3

Strong Distributivity

$<\kappa$-closed forcings have some nice regularity properties which in general do not hold for forcings which are merely $<\kappa$-distributive (e.g. preserving stationarity in κ). In this chapter, we will introduce a strengthening of $<\kappa$-distributivity, which $<\kappa$-closure turns into after forcing with a κ-c.c. forcing and show that it can replace $<\kappa$-closure in some important applications. This concept is due to the author (mostly from [Jak23]).

3.1 Definition and Basic Results

Definition 3.1.1. A notion of forcing $\mathbb{P}$ is *strongly $<\kappa$-distributive* if for any sequence $(D_\alpha)_{\alpha<\kappa}$ of open dense sets and any $p \in \mathbb{P}$, there is a descending sequence $(p_\alpha)_{\alpha<\kappa}$ such that $p_0 \leq p$ and $\forall \alpha < \kappa$, $p_\alpha \in D_\alpha$. Such a sequence will be called a *thread* through $(D_\alpha)_{\alpha<\kappa}$.

Strong $<\kappa$-distributivity can be thought of as having $<\kappa$-distributivity witnessed in a uniform way: If $(D_\alpha)_{\alpha<\kappa}$ is a sequence of open dense subsets of some $<\kappa$-distributive forcing notion, there is a sequence $(p_\alpha)_{\alpha<\kappa}$ such that for all $\alpha < \kappa$, $p_\alpha \leq p_0$ and $p_\alpha \in \bigcap_{\beta<\alpha} D_\beta$ (since the intersection of $<\kappa$ open dense sets is open dense). However, we cannot in general find such a sequence in a uniform way, i.e. such that it is descending.

Obviously strong $<\kappa$-distributivity implies $<\kappa$-distributivity. Note that strong $<\kappa$-distributivity and $<\kappa$-distributivity are not equivalent: If $S \subsetneq \omega_1$ is stationary and co-stationary, the forcing shooting a club through S by countable approximations is $<\omega_1$-distributive (cf. [Jec03, Lemma 23.9]). However, as we will later show, it cannot be strongly $<\omega_1$-distributive as it destroys the stationarity of a subset of ω_1.

Keeping with the theme of strong $<\kappa$-distributivity being a uniform version of $<\kappa$-distributivity, we have the following characterisation of strong $<\kappa$-distributivity: Recall that for antichains A, B we say that A *refines* B if for every $q \in A$ there is $q' \in B$ with $q \leq q'$.

Lemma 3.1.2. *For a forcing order $\mathbb{P}$, the following are equivalent:*

1. *$\mathbb{P}$ is strongly $<\kappa$-distributive.*
2. *$\mathbb{P}$ is $<\kappa$-distributive and for $p \in \mathbb{P}$ and any descending sequence $(A_\alpha)_{\alpha<\kappa}$ (with regards to refinement) of maximal antichains below p, there is a descending sequence $(p_\alpha)_{\alpha<\kappa}$ such that $p_0 \leq p$ and for any α, $p_\alpha \in A_\alpha$.*

Proof. Assume $\mathbb{P}$ is strongly $<\kappa$-distributive. Of course, this implies that $\mathbb{P}$ is $<\kappa$-distributive. Let $(A_\alpha)_{\alpha<\kappa}$ be a sequence of maximal antichains in $\mathbb{P}$ such that for $\beta < \alpha$, A_α refines A_β. For $\alpha < \kappa$, let D_α be the downward closure of A_α and consider a thread $(q_\alpha)_{\alpha<\kappa}$ through $(D_\alpha)_{\alpha<\kappa}$. For any $\alpha < \kappa$, let p_α be the unique (by pairwise incompatibility) element of A_α that is above q_α. We are done after showing

Claim. *The sequence $(p_\alpha)_{\alpha<\kappa}$ is descending.*

Proof. Let $\beta < \alpha$ be arbitrary. Because A_α refines A_β, there exists p'_β such that $p_\alpha \leq p'_\beta$. Thus, $q_\alpha \leq p_\alpha \leq p'_\beta$ and $q_\alpha \leq q_\beta \leq p_\beta$. In summary, p'_β and p_β are compatible and therefore equal, which implies $p_\alpha \leq p_\beta$. $\square$

Now assume condition (2) holds. Let $(D_\alpha)_{\alpha<\kappa}$ be a sequence of open dense subsets of $\mathbb{P}$. Inductively, and using $<\kappa$-distributivity, construct a sequence $(A_\alpha)_{\alpha<\kappa}$ such that $A_\alpha \subseteq D_\alpha$ is a maximal antichain and for $\beta < \alpha$, A_α refines A_β. It follows that a thread through $(A_\alpha)_{\alpha<\kappa}$ is also one through $(D_\alpha)_{\alpha<\kappa}$. $\square$

While $<\kappa$-distributivity means that every $<\kappa$-sequence of ground-model elements is in the ground model, strong $<\kappa$-distributivity means that we can uniformly approximate even κ-sequences of ground-model elements.

Lemma 3.1.3. *Assume $\mathbb{P}$ is strongly $<\kappa$-distributive, $p \in \mathbb{P}$ and $\dot{f}$ is a $\mathbb{P}$-name such that $p \Vdash \dot{f} : \check{\kappa} \to V$. Then there is a descending sequence $(p_\alpha)_{\alpha<\kappa}$ with $p_0 \leq p$ such that for every $\alpha < \kappa$, p_α decides $\dot{f}(\check{\alpha})$.*

Proof. Consider $D_\alpha := \{q \in \mathbb{P} \mid q \text{ decides } \dot{f}(\check{\alpha})\}$. Clearly a thread through $(D_\alpha)_{\alpha<\kappa}$ is as required. $\square$

As is the case for $<\kappa$-distributivity, the converse holds for separative forcing orders (but we will never use this).

3.2 A Game Characterisation of Strong Distributivity

In this section we obtain a new version of Foreman's Theorem from [For83], relating strong $<\kappa$-distributivity to the non-existence of a winning strategy for INC in the completeness game.

Theorem 3.2.1 (Foreman)**.** *Let κ be a cardinal and $\mathbb{P}$ a forcing order. $\mathbb{P}$ is $<\kappa^+$-distributive if and only if INC does not have a winning strategy in $G(\mathbb{P}, \kappa + 1)$.*

If INC does not have a winning strategy in $G(\mathbb{P}, \lambda + 1)$, they do not have one in any $G(\mathbb{P}, \mu)$ for $\mu < \lambda^+$. Having this witnessed uniformly suggests the following statement:

Theorem 3.2.2. $\mathbb{P}$ *is strongly $<\kappa$-distributive if and only if INC does not have a winning strategy in $G(\mathbb{P}, \kappa)$.*

Proof. For one direction, if $(D_\alpha)_{\alpha<\kappa}$ is a sequence of open dense sets without a thread below some $p \in \mathbb{P}$, let INC first play p and then, after $(p_\delta)_{\delta<\gamma+2n+1}$ has been played, a condition $p_{\gamma+2n+1} \in D_{\gamma+n}$ with $p_{\gamma+2n+1} \leq p_{\gamma+2n}$. It is clear that this strategy wins for INC (otherwise, a losing game for INC would give rise to a thread through $(D_\alpha)_{\alpha<\kappa}$ by picking out the odd conditions).

For the other direction, let σ be a winning strategy for INC in $G(\mathbb{P}, \kappa)$. Let $\sigma(1_{\mathbb{P}}) = p$. We will construct a sequence $(A_\alpha)_{\alpha\in\kappa}$ such that the following holds:

1. For each $\alpha \in \kappa$ and $p_\alpha \in A_\alpha$, there exists a unique sequence $(p_\beta)_{\beta<\alpha}$ such that for all $\beta \leq \alpha$, $p_\beta \in A_\beta$ and if $\beta \leq \alpha$ is odd, $p_\beta = \sigma((p_\delta)_{\delta<\beta})$.
2. If $\alpha \in \kappa$ is odd, A_α is a maximal antichain below p (for even α, we carefully choose A_α to obtain uniqueness in (1)).

To begin, let $A_0 := \{1_{\mathbb{P}}\}$ and $A_1 := \{p\}$. Assume the sequence has been constructed until some even successor ordinal γ. We will construct A_γ and $A_{\gamma+1}$ simultaneously. Let $D_{\gamma+1}$ consist of all $p' \in \mathbb{P}$ such that there exists a sequence $(p_\alpha)_{\alpha<\gamma+2}$ with $p_{\gamma+1} = p'$ such that for all $\alpha < \gamma$, $p_\alpha \in A_\alpha$ and if α is odd, $p_\alpha = \sigma((p_\beta)_{\beta<\alpha})$.

Claim. *$D_{\gamma+1}$ is dense below p.*

Proof. Let $p' \leq p$ be arbitrary. By maximality of $A_{\gamma-1}$, there exists $p_{\gamma-1} \in A_{\gamma-1}$ compatible with p', witnessed by some p''. By the inductive hypothesis, there exists a unique sequence $\overline{p} = (p_\beta)_{\beta<\gamma-1}$ with $p_\beta \in A_\beta$ for $\beta < \gamma - 1$ and $p_\beta = \sigma((p_\delta)_{\delta<\beta})$ for all odd $\beta \leq \gamma - 1$. Hence, letting $s := \overline{p}^\frown p_{\gamma-1}{}^\frown p''$, $s^\frown\sigma(s)$ witnesses density, since $\sigma(s) \leq p'' \leq p'$. $\square$

Let $A_{\gamma+1} \subseteq D_{\gamma+1}$ be a maximal antichain below p. For each $p_{\gamma+1} \in A_{\gamma+1}$, by the definition of $D_{\gamma+1}$, there exists a sequence $(p_\alpha)_{\alpha<\gamma+1}$ such that for all $\alpha \leq \gamma + 1$, $p_\alpha \in A_\alpha$ and if $\alpha \leq \gamma + 1$ is odd, $p_\alpha = \sigma((p_\beta)_{\beta<\alpha})$. Choose such a sequence for each $p_{\gamma+1} \in A_{\gamma+1}$ and let A_γ consist of the γth entries of these sequences.

Claim. *For each $p_{\gamma+1} \in A_{\gamma+1}$, there exists a unique sequence $(p_\beta)_{\beta<\gamma+1}$ such that for all $\beta \leq \gamma + 1$, $p_\beta \in A_\beta$ and if $\beta \leq \gamma + 1$ is odd, $p_\beta = \sigma((p_\delta)_{\delta<\beta})$*

Proof. Let $p_{\gamma+1} \in A_{\gamma+1}$ and let s be the chosen sequence witnessing $p_{\gamma+1} \in D_{\gamma+1}$. We will verify that any sequence as above is equal to s. So let $s' = (p'_\beta)_{\beta\leq\gamma+1}$ be a different sequence such that for all $\beta \leq \gamma+1$, $p'_\beta \in A_\beta$ and if $\beta \leq \gamma+1$ is odd, $p'_\beta = \sigma((p'_\delta)_{\delta<\beta})$ with $p_{\gamma+1} = p'_{\gamma+1}$.

It follows that $p'_{\gamma-1}$ and $p_{\gamma-1}$ are compatible (witnessed by $p_{\gamma+1} = p'_{\gamma+1}$) and thus equal, as $A_{\gamma-1}$ is an antichain. By the inductive hypothesis $s \restriction \gamma = s' \restriction \gamma$, so $p_\gamma \neq p'_\gamma$. Since $p'_\gamma \in A_\gamma$ there is $a_{\gamma+1} \in A_{\gamma+1}$ (necessarily different from $p_{\gamma+1}$) and a sequence t witnessing $a_{\gamma+1} \in D_{\gamma+1}$ such that $t(\gamma) = p'_\gamma$. Then $t(\gamma - 1)$ and $p'_{\gamma-1}$ are compatible (witnessed by $p'_\gamma = t(\gamma)$) and hence equal. As before, this implies $t \restriction \gamma = s' \restriction \gamma$. However, this means that

$$\sigma((p'_\delta)_{\delta<\gamma+1}) = p'_{\gamma+1} = p_{\gamma+1} \neq a_{\gamma+1} = \sigma(t \restriction \gamma + 1)$$

contradicting the fact that σ is a function and $(p'_\delta)_{\delta<\gamma+1} = t \restriction \gamma + 1$. $\square$

Assume γ is a limit. Let A'_γ be a common refinement of A_α for odd $\alpha < \gamma$. Given $p \in A'_\gamma$, let $p_\alpha \in A_\alpha$ witness refinement for odd α and let $p_\alpha \in A_\alpha$ witness $p_{\alpha+1} \in A_{\alpha+1}$ for even α. Then $(p_\alpha)_{\alpha<\gamma}$ is a play according to σ by uniqueness (which implies that the sequences witnessing $p_\alpha \in A_\alpha$ are coherent). Let D_γ be the downward closure of A'_γ and let $D_{\gamma+1}$ consist of $\sigma(s)$ for sequences $s = (p_\alpha)_{\alpha<\gamma+1}$ with $p_\gamma \in D_\gamma$ and $s \restriction \gamma$ witnessing this. Thus, $D_{\gamma+1}$ is dense and we can proceed as in the previous step: Let $A_{\gamma+1} \subseteq D_{\gamma+1}$ be a maximal antichain. Let $A_\gamma \subseteq D_\gamma$ contain one witness to $p \in D_{\gamma+1}$ for each $p \in A_{\gamma+1}$. Then clearly for any $p \in A_{\gamma+1}$ a sequence as claimed exists. As before, if there exist two sequences t, t' for one $p \in A_{\gamma+1}$, $t \restriction \gamma = t' \restriction \gamma$, since $t(\gamma) \in A_\gamma \subseteq A_{\gamma'}$ and thus lies below exactly one element of each A_α for odd α.

Lastly, there exists a thread through $(A_\alpha)_{\alpha\in\kappa\cap\mathrm{Odd}}$, i.e. a sequence $(p_\alpha)_{\alpha\in\kappa\cap\mathrm{Odd}}$ such that for odd α, $p_\alpha \in A_\alpha$. For even α, let $p_\alpha \in A_\alpha$ witness $p_{\alpha+1} \in A_{\alpha+1}$. By uniqueness, $(p_\alpha)_{\alpha<\kappa}$ is a play in $G(\mathbb{P}, \kappa)$ according to σ. But this contradicts our assumption that σ was a winning strategy. □

Thus, just like $<\kappa$-strategic closure, strong $<\kappa$-distributivity is situated between $<\kappa$-distributivity and κ-strategic closure. We will show later that, in general, strong $<\kappa$-distributivity neither implies $<\kappa$-strategic closure nor is implied by it.

3.3 The Strong Easton Lemma and Stationary Preservation

The Easton lemma was used by William Easton in his famous proof that whenever F is a function mapping regular cardinals to regular cardinals such that F is increasing and $\mathrm{cf}(F(\delta)) > \delta$ for any δ, there is a model of ZFC where for all regular δ, $2^\delta = F(\delta)$. It states the following:

Lemma 3.3.1 (Easton Lemma)**.** *Let κ be a regular cardinal. Assume $\mathbb{P}$ is κ-c.c. and $\mathbb{Q}$ is $<\kappa$-closed. Then:*

1. $1_{\mathbb{Q}} \Vdash \check{\mathbb{P}}$ *is κ-c.c.*
2. $1_{\mathbb{P}} \Vdash \check{\mathbb{Q}}$ *is $<\check{\kappa}$-distributive.*

Using the concept of strong distributivity, we can improve this statement in two directions, weakening the assumptions and strengthening the conclusions. Moreover, this makes the lemma "symmetrical". We note that a statement similar to (2) was shown by Lietz on Mathoverflow (see [Lie]) following a question posed by the author.

Lemma 3.3.2. *Let κ be a regular cardinal. Assume $\mathbb{P}$ is κ-c.c. and $\mathbb{Q}$ is strongly $<\kappa$-distributive.*

1. $1_{\mathbb{Q}} \Vdash \check{\mathbb{P}}$ *is $\check{\kappa}$-c.c.*
2. $1_{\mathbb{P}} \Vdash \check{\mathbb{Q}}$ *is strongly $<\check{\kappa}$-distributive.*

Proof. We show the statements one after the other:

1. Assume $\dot{f}$ is forced by some $q \in \mathbb{Q}$ to be an enumeration of an antichain in $\mathbb{P}$ of size κ. Thus, $q \Vdash \dot{f} \colon \check{\kappa} \to V$. Hence, there exists a descending sequence $(q_\alpha)_{\alpha<\kappa}$ with $q_0 \leq q$ such that $q_\alpha \Vdash \dot{f}(\check{\alpha}) = \check{p}_\alpha$ for some p_α.

   Claim. *$\{p_\alpha \mid \alpha < \kappa\}$ is an antichain in $\mathbb{P}$.*

   Proof. Let $\beta < \alpha < \kappa$ be arbitrary. Then $q_\alpha \Vdash \dot{f}(\check{\alpha}) = \check{p}_\alpha \wedge \dot{f}(\check{\beta}) = \check{p}_\beta$. Because $q_\alpha \leq q_0$, $q_\alpha \Vdash \check{p}_\alpha \perp \check{p}_\beta$, ergo $p_\alpha \perp p_\beta$. □

   This claim directly contradicts our assumption that $\mathbb{P}$ was κ-c.c.

2. We first show a helpful claim

   Claim. *If $D \subseteq \mathbb{P} \times \mathbb{Q}$ is open dense and $p \in \mathbb{P}$, the set $D_p \subseteq \mathbb{Q}$ consisting of all $q \in \mathbb{Q}$ such that for some $A \subseteq \mathbb{P}$ that is a maximal antichain below p, $A \times \{q\} \subseteq D$, is open dense in $\mathbb{Q}$.*

   Proof. Openness is clear: If A witnesses $q \in D_p$ and $q' \leq q$, A also witnesses $q' \in D_p$.

   Thus, assume the set is not dense and there is $q \in \mathbb{Q}$ such that for every $q' \leq q$, $q' \notin D_p$. We will give a winning strategy for INC in $G(\mathbb{P}, \kappa)$. In every run $(q_\alpha)_{\alpha<\gamma}$ of the game, we construct an antichain $\{p_\alpha \mid \alpha \in \gamma \cap \mathrm{Odd}\}$ below p such that $(p_\alpha, q_\alpha) \in D$. To begin, let INC find a pair $(p_1, q_1) \leq (p, q)$ with $(p_1, q_1) \in D$ and play q_1.

   Assume the game has lasted until γ, $\gamma + 1$ is odd and COM has just played q_γ. If $\{p_\alpha \mid \alpha \in \gamma \cap \mathrm{Odd}\}$ is maximal below p, it witnesses $q_\gamma \in D_p$ by openness of D: For every $\alpha \in \gamma \cap \mathrm{Odd}$, $(p_\alpha, q_\alpha) \in D$ and thus $(p_\alpha, q_\gamma) \in D$. This contradicts our assumption, since $q_\gamma \leq q$. It follows that there exists some $p'_{\gamma+1}$ which is incompatible with every p_α. By open density, there exists $(p_{\gamma+1}, q_{\gamma+1}) \leq (p'_{\gamma+1}, q_\gamma)$, $(p_{\gamma+1}, q_{\gamma+1}) \in D$. Let INC play $q_{\gamma+1}$.

   This strategy is a winning strategy, because a play of length κ would give us a κ-sized antichain in $\mathbb{P}$. This contradicts our assumptions. □

   Now assume $\dot{f}$ and τ are $\mathbb{P}$-names such that $\dot{f}$ is forced by some p to map $\check{\kappa}$ to open dense subsets of $\mathbb{Q}$ and τ to be an element of $\mathbb{Q}$. Strengthening p if necessary, we can assume $p \Vdash \tau = \check{q}$ for some $q \in \mathbb{Q}$.

   Claim. *The set $D_\alpha := \{(p', q') \in \mathbb{P} \restriction p \times \mathbb{Q} \mid p' \Vdash \check{q}' \in \dot{f}(\check{\alpha})\}$ is open dense.*

   Proof. Openness in both coordinates follows either from the properties of the forcing relation or from $\dot{f}(\check{\alpha})$ being forced by p to be open.

   For density, let $(p', q') \in \mathbb{P} \restriction p \times \mathbb{Q}$ be arbitrary. Thus $p' \Vdash \exists \tau (\tau \in \dot{f}(\check{\alpha}) \wedge \tau \leq \check{q}')$. Because τ is in particular forced to be in V, there exists $p'' \leq p'$ and q'' such that

   $$p'' \Vdash (\check{q}'' \in \dot{f}(\check{\alpha}) \wedge \check{q}'' \leq \check{q}')$$

   Thus, $(p'', q'') \leq (p', q')$ and $(p'', q'') \in D_\alpha$ □

Combining the two claims, for each α, the set E_α, consisting of all $q' \in \mathbb{Q}$ such that for some $A \subseteq \mathbb{P}$ which is a maximal antichain below p we have $A \times \{q'\} \subseteq D_\alpha$, is open dense in $\mathbb{Q}$. If $q' \in E_\alpha$, there exists a maximal antichain A below p such that for every $p' \in A$, $p' \Vdash \check{q}' \in \dot{f}(\check{\alpha})$. By maximality, $p \Vdash \check{q}' \in \dot{f}(\check{\alpha})$.

Let $(q_\alpha)_{\alpha<\kappa}$ be a thread through $(E_\alpha)_{\alpha<\kappa}$ below q. Then p forces $(\check{q}_\alpha)_{\alpha<\kappa}$ to be a thread through $\dot{f}$ below $\check{q}$.

$\square$

We note that in the previous lemma the assumption of strong $<\kappa$-distributivity cannot be relaxed to mere $<\kappa$-distributivity: It is consistent that there is a Suslin tree T such that T^2 collapses ω_1. Ergo, despite T being both c.c.c. and $<\omega_1$-distributive, T is neither c.c.c. nor $<\omega_1$-distributive in any extension by T.

As stated before, strongly $<\kappa$-distributive forcings preserve stationary subsets of κ (contrasted by the fact that we can destroy stationary subsets of ω_1 using a $<\omega_1$-distributive forcing notion):

Lemma 3.3.3. *If $\mathbb{P}$ is strongly $<\kappa$-distributive and $S \subseteq \kappa$ is stationary, $1_{\mathbb{P}} \Vdash \check{S}$ is stationary.*

Proof. Assume that some $p \in \mathbb{P}$ forces $\dot{C} \subseteq \check{\kappa}$ to be a club and $\dot{f}$ to be its strictly increasing enumeration. Thus, $p \Vdash \dot{f}\colon \check{\kappa} \to V$. Hence, there exists a descending sequence $(p_\alpha)_{\alpha<\kappa}$ such that $p_\alpha \Vdash \dot{f}(\check{\alpha}) = \check{\gamma}_\alpha$ for some $\gamma_\alpha \in \kappa$. Let

$$C' := \{\gamma_\alpha \mid \alpha < \kappa\}$$

Claim. *$C' \subseteq \kappa$ is club.*

Proof. If $\beta < \alpha < \kappa$, $p_\alpha \Vdash \check{\gamma}_\beta = \dot{f}(\check{\beta}) \wedge \check{\gamma}_\alpha = \dot{f}(\check{\alpha})$. Because $\dot{f}$ is forced to be strictly increasing, $p_\alpha \Vdash \check{\gamma}_\beta < \check{\gamma}_\alpha$. Thus $(\gamma_\alpha)_{\alpha<\kappa}$ is a strictly increasing sequence in κ of length κ and so C' is unbounded in κ.

Let $\gamma \in \kappa$ be a limit and assume $C' \cap \gamma = \{\gamma_\alpha \mid \alpha < \delta\}$ is unbounded in γ. This implies that δ is a limit ordinal (because the sequence is strictly increasing). Thus, p_δ forces that $(\dot{f}(\check{\alpha}))_{\alpha<\delta}$ is unbounded in γ. Because p_δ forces $\dot{f}(\check{\delta}) = \gamma_\delta$ and $\dot{f}$ to be continuous, $\gamma_\delta = \gamma \in C'$. $\square$

Since $C' \subseteq \kappa$ is club and in V, there exists $\alpha < \kappa$ such that $\gamma_\alpha \in C' \cap S$. Then $p_\alpha \Vdash \check{\gamma}_\alpha \in \check{S} \cap \dot{C}$. $\square$

Since clubs in κ are basically the same as clubs in $[\kappa]^{<\kappa}$, we have the following corollary:

Corollary 3.3.4. *If $\mathbb{P}$ is strongly $<\kappa$-distributive, $\mathbb{P}$ preserves stationary subsets of $[X]^{<\kappa}$ whenever $|X| = \kappa$.*

Proof. We will show that for every set $S \subseteq [X]^{<\kappa}$, there exists a set $S' \subseteq \kappa$ such that S is stationary if and only if S' is and this remains the case in every extension of V.

Let $F\colon X \to \kappa$ be a bijection. Then $a \mapsto F[a]$ is a continuous and order-preserving bijection from $[X]^{<\kappa}$ into $[\kappa]^{<\kappa}$. Hence, S is stationary if and only if $F[[S]] := \{F[a] \mid a \in S\}$ is. Let $S' := F[[S]] \cap \kappa$.

Claim. *$S' \subseteq \kappa$ is stationary if and only if $F[[S]] \subseteq [\kappa]^{<\kappa}$ is.*

Proof. Assume S' is nonstationary and let $C \subseteq \kappa$ be a club with empty intersection with S'. By standard arguments, $C \subseteq [\kappa]^{<\kappa}$ is also club. Since $C \cap F[[S]] = \emptyset$, $F[[S]]$ is nonstationary.

Assume $F[[S]]$ is nonstationary and let $C \subseteq [\kappa]^{<\kappa}$ be club with empty intersection with $F[[S]]$. It follows that $C \cap \kappa \subseteq \kappa$ is also club. Since $(C \cap \kappa) \cap S' = \emptyset$, we are done. ☐

Ergo if $\mathbb{P}$ destroys the stationarity of S, it destroys the stationarity of $F[[S]]$ and thus of S', a contradiction. ☐

3.4 Very Fat Sets

This small section which is tangential to the remainder of the thesis investigates the notion of *very fat sets* which are connected to strong distributivity.

If δ is any ordinal with uncountable cofinality, a subset $S \subseteq \delta$ is called *fat* if for any club $C \subseteq \delta$ and any $\xi < \delta$, $S \cap C$ contains a closed subset of ordertype ξ. In [Kru08c], Krueger generalized this definition to subsets of $[X]^{<\kappa}$ as follows:

Definition 3.4.1. Suppose κ is a regular uncountable cardinal and $\kappa \subseteq X$. A set $A \subseteq [X]^{<\kappa}$ is *fat* if for all regular $\Theta \geq \kappa$ with $X \subseteq H(\Theta)$, any club $C \subseteq [H(\Theta)]^{<\kappa}$ and any $\xi < \kappa$ there is an increasing and continuous sequence $(N_i)_{i<\xi}$ such that for all $i < \xi$, $N_i \in C$, $N_i \cap X \in A$ and $N_i \in N_{i+1}$ provided $i + 1 < \xi$.

Fatness for subsets of ordinals is related to these sets being able to obtain clubs through distributive forcing extensions:

Lemma 3.4.2 (Abraham and Shelah, [AS83])**.** *Suppose κ is strongly inaccessible or $\kappa = \mu^+$ where $\mu^{<\mu} = \mu$. Let $A \subseteq \kappa$. The following are equivalent:*

1. *A is fat.*
2. *There is a $<\kappa$-distributive forcing poset which forces that A contains a club.*

The forcing notion simply consists of increasing and continuous sequences of elements of A, ordered by end-extension. Krueger's generalized notion of fatness has the same characterization:

Lemma 3.4.3 (Krueger, [Kru08c], Theorem 2.4)**.** *Suppose κ is strongly inaccessible or $\kappa = \mu^+$ where $\mu^{<\mu} = \mu$. Let X be a set containing κ and $A \subseteq [X]^{<\kappa}$. The following are equivalent:*

1. *A is fat.*

2. *There is a $< \kappa$-distributive forcing poset which forces that there is an increasing and continuous sequence $(a_i)_{i<\kappa}$ which is cofinal in $[X]^{<\kappa}$ such that $a_i \in A$ for $i < \kappa$.*

As in the lemma by Abraham and Shelah, Krueger's poset simply consists of increasing and continuous sequences of elements of A, ordered by end-extension.

We define a strengthening of Krueger's notion of fatness which has a similar characterisation using strong distributivity:

Definition 3.4.4. Suppose κ is a regular uncountable cardinal and $\kappa \subseteq X$. A set $A \subseteq [X]^{<\kappa}$ is *very fat* if for all regular $\Theta \geq \kappa$ with $X \subseteq H(\Theta)$ and any club $C \subseteq [H(\Theta)]^{<\kappa}$ there is an increasing and continuous sequence $(N_i)_{i<\kappa}$ such that for all $i < \kappa$, $N_i \in C$, $N_i \cap X \in A$ and for all $j < \kappa$, $(N_i)_{i \leq j} \in N_{j+1}$.

We modified the previous definition slightly (requiring that N_{j+1} does not just contain N_j but actually the whole sequence $(N_i)_{i \leq j}$) for technical reasons: To show $< \mu^+$-distributivity, it suffices to show that intersections of μ many open dense sets are open dense. This is mostly done by constructing a descending sequence of length μ. All of the initial segments of this sequence of course have length $< \mu$. In our case however, we need to construct a sequence of length μ^+ which therefore has initial segments of length $\geq \mu$.

We note that for $|X| = \kappa$, a set $A \subseteq [X]^{<\kappa}$ is very fat if and only if it contains a club in $[X]^{<\kappa}$: Clearly, if A contains a club, A is very fat. On the other hand, assume A is very fat and let $F \colon X \to \kappa$ be a bijection. Let $C \subseteq [H(\Theta)]^{<\kappa}$ consist of all those $M \prec H(\Theta)$ with $F \in M$ (letting Θ be large enough). If $(N_i)_{i<\kappa}$ is as required, we of course have $\kappa \subseteq \bigcup_{i<\kappa} N_i$, so by elementarity $X \subseteq \bigcup_{i<\kappa} N_i$. In particular, $\{N_i \cap X \mid i < \kappa\} \subseteq A$ and is club in $[X]^{<\kappa}$.

However, for $|X| > \kappa$, this notion is not equivalent to clubness, as we will later see.

We first show an analogue of Lemmas 3.4.2 and 3.4.3.

Lemma 3.4.5. *Suppose κ is strongly inaccessible or $\kappa = \mu^+$ where $\mu^{<\mu} = \mu$. Let X be a set containing κ and $A \subseteq [X]^{<\kappa}$. The following are equivalent:*

1. *A is very fat.*

2. *There is a strongly $< \kappa$-distributive forcing poset which forces that there is an increasing and continuous sequence $(a_i)_{i<\kappa}$ which is cofinal in $[X]^{<\kappa}$ such that $a_i \in A$ for $i < \kappa$.*

Proof. First suppose A is very fat. Let $\mathbb{P}$ consist of increasing and continuous sequences $(a_i)_{i<\xi}$ for $\xi < \kappa$ a successor ordinal, ordered by end-extension.

Claim. *$\mathbb{P}$ is strongly $< \kappa$-distributive.*

Proof. Let $\overline{D} := (D_i)_{i<\kappa}$ be a sequence of open dense subsets of $\mathbb{P}$. Let Θ be sufficiently large and $C \subseteq [H(\Theta)]^{<\kappa}$ the club of all $M \prec H(\Theta)$ such that $\mathbb{P}, A, X, \overline{D} \in M$. By induction on $i < \kappa$ we define a descending sequence $(p_i)_{i<\kappa}$ such that $p_i \in N_{i+1}$ $p_i \in \bigcap_{j<i} D_j$ and $\max(p_i) = N_i \cap X$. Suppose p_i has been defined for all $i < j$.

Assume j is a successor ordinal, $j = i + 1$. Let $p_j^* \leq p_i$ such that $p_j \in D_i$ (clearly $i \in j \subseteq N_j \cap \kappa$) and $p_j \in N_j$ (this is possible as $p_i \in N_j$ by assumption). Then define $p_j := p_j \cup \{N_j \cap X\}$ which is as required.

If j is a limit ordinal, simply let $p_j := \left(\bigcup_{i<j} p_i\right) \cup \{N_j \cap X\}$. Clearly p_j is a condition in $\mathbb{P}$ and $p_j \in \bigcap_{i<j} D_i$. Lastly, $p_j \in N_{j+1}$ because it is defined using $(N_i)_{i\leq j}$ and this sequence is in N_{j+1}. Clearly the sequence $(p_{i+1})_{i<\kappa}$ is as required. $\square$

The preceding claim implies that $\{p \in \mathbb{P} \mid \alpha \in \mathrm{dom}(p)\}$ is dense for every $\alpha < \kappa$. Ergo, if G is $\mathbb{P}$-generic, $\bigcup G$ is an increasing and continuous sequence $(a_i)_{i<\kappa}$ which is cofinal in $[X]^{<\kappa}$ (by genericity) and consists of members of A.

Now assume $\mathbb{P}$ is a strongly $<\kappa$-distributive forcing poset and G is $\mathbb{P}$-generic such that in $V[G]$ there is an increasing and continuous sequence $(a_i)_{i<\kappa}$ which is cofinal in $[X]^{<\kappa}$ such that $a_i \in A$ for $i < \kappa$. We show that A is very fat. To this end, let Θ be large and $C \subseteq [H(\Theta)]^{<\kappa}$ club. By the distributivity, C is still club in $[H^V(\Theta)]^{<\kappa}$ in $V[G]$.

Claim. *In $V[G]$, there exists an increasing and continuous sequence $(N_i)_{i<\kappa}$ such that for all $i < \kappa$, $N_i \in C$, $N_i \cap X \in A$ and $(N_j)_{j\leq i} \in N_{i+1}$.*

Proof. Such a sequence is easily definable by induction on $i < \kappa$: By making sure $N_i \cap X$ is always taken from $\{a_j \mid j < \kappa\}$, we can go through the limit step. $\square$

In V, let $(\dot{N}_i)_{i<\kappa}$ be a sequence of $\mathbb{P}$-names for the elements of the sequence. By strong distributivity of $\mathbb{P}$, there is a function f in V and a descending sequence $(p_i)_{i<\kappa}$ of elements of $\mathbb{P}$ such that p_i forces $\check{f(i)} = \dot{N}_i$ (since any $\dot{N}_i$ is forced to be in V by $<\kappa$-distributivity). However, this clearly implies that the sequence $(f(i))_{i<\kappa}$ witnesses that A is very fat (with respect to Θ and C), since for any i, p_{i+1} forces that $\check{f(i)} = \dot{N}_i \in \check{C}$ (so $f(i) \in C$), $\check{f(i)} \cap \check{X} = \dot{N}_i \cap \check{X} \in \check{A}$ (so $f(i) \cap X \in A$) and lastly $(\check{f(j)})_{j\leq i} = (\dot{N}_j)_{j\leq i} \in \dot{N}_{i+1} = \check{f(i+1)}$, so $(f(j))_{j\leq i} \in f(i+1)$. $\square$

This provides an additional proof to the assertion that all very fat subsets $A \subseteq [X]^{<\kappa}$ contain a club if $|X| = \kappa$: Any forcing adding a club to A destroys the would-be stationarity of $[X]^{<\kappa} \setminus A$ which it cannot do if it is strongly $<\kappa$-distributive.

This raises the natural question whether there can exist very fat subsets of $[X]^{<\kappa}$ which do not contain a club, provided $|X| > \kappa$. We first note that Krueger's proof of Proposition 3.1 in [Kru08c] also shows that $\mathbb{P}$ forces that $[X]^{<\kappa} \cap V$ is very fat in $V[\mathbb{P}]$ whenever $\mathbb{P}$ is κ-c.c. The other missing ingredient is the following theorem:

Theorem 3.4.6 ([AS83], Theorem 7)**.** *Assume $\mathbb{P}$ is c.c.c. and adds a real. Then $\mathbb{P}$ forces that $[\omega_2]^{<\omega_1} \setminus V$ is stationary in $[\omega_2]^{<\omega_1}$.*

This theorem was refined both by Gitik in [Git85] and Krueger in [Kru09]. Gitik showed that it is only necessary that $\mathbb{P}$ adds a real and does not collapse ω_1 while Krueger showed that the added stationary set has a certain structure which implies that it is preserved by countably closed forcings. We will use Krueger's refinement in chapter 7.

Combining all of these results, we see immediately:

Theorem 3.4.7. *After forcing with* $\mathrm{Add}(\omega)$*, the set* $[\omega_2]^{<\omega_1} \setminus V$ *is very fat but does not contain a club subset of* $[\omega_2]^{<\omega_1}$.

Different results in this direction come from the so-called *strong reflection principle* which is a powerful consequence of Martin's Maximum. We use the following equivalent version due to Feng and Jech (see [FJ98]) called the *projective stationary reflection principle*:

Definition 3.4.8. Let $\Theta \geq \omega_2$. $S \subseteq [H(\Theta)]^{<\omega_1}$ is *projective stationary* if for any stationary $A \subseteq \omega_1$, the set

$$S(A) := \{x \in S \mid x \cap \omega_1 \in A\}$$

is stationary in $[H(\Theta)]^{<\omega_1}$.

The *projective stationary reflection principle* SRP states that whenever $S \subseteq [H(\Theta)]^{<\omega_1}$ is projective stationary, there exists a continuous sequence $(N_i)_{i<\omega_1}$ of elements of S such that $(N_i)_{i\leq j} \in N_{j+1}$ for any $j < \omega_1$.

So SRP implies that any projective stationary set is very fat: Let $S \subseteq [H(\Theta)]^{<\omega_1}$ be projective stationary. Let Θ' be large enough and $C \subseteq [H(\Theta')]^{<\omega_1}$ club. Let S' consist of those $M \in [H(\Theta')]^{<\omega_1}$ such that $M \in C$ and $M \cap H(\Theta) \in S$. Given any stationary $A \subseteq \omega_1$, the set $S(A)$ is stationary in $[H(\Theta)]^{<\omega_1}$, so $\{M \in C \mid M \cap H(\Theta) \in S(A)\}$ is stationary in $[H(\Theta')]^{<\omega_1}$. Ergo S' is projective stationary and there exists a sequence as desired by SRP.

The principle SRP enables us to show the following:

Proposition 3.4.9. *It is consistent that there exist two disjoint very fat subsets of* $[H(\Theta)]^{<\omega_1}$.

Proof. Assume SRP. Partition $E^{\omega_2}_{\omega} := \{\alpha \in \omega_2 \mid \mathrm{cf}(\alpha) = \omega\}$ into two disjoint stationary sets A and B. Let $S(A)$ consist of those $M \in [H(\Theta)]^{<\omega_1}$ such that $\sup(M \cap \omega_2) \in A$ and define $S(B)$ the same way. Clearly $S(A)$ and $S(B)$ are disjoint. By arguments in [FJ98], both are projective stationary and so they are very fat by SRP. □

This implies the following:

Proposition 3.4.10. *It is consistent that there are strongly* $<\omega_1$*-distributive forcing notions* $\mathbb{P}$ *and* $\mathbb{Q}$ *such that* $\mathbb{P} \times \mathbb{Q}$ *is not* $<\omega_1$*-distributive.*

Proof. Let A and B be disjoint very fat subsets of $[H(\omega_2)]^{<\omega_1}$. The forcing notions $\mathbb{P}$ and $\mathbb{Q}$ shooting a continuous, $\in$-increasing cofinal ω_1-sequence through A and B respectively are strongly $<\omega_1$-distributive. However, in any extension by $\mathbb{P} \times \mathbb{Q}$ there are two disjoint club subsets of $([H(\omega_2)]^{<\omega_1})^V$. The only possibility for this is if $\mathbb{P} \times \mathbb{Q}$ collapses ω_1^V. □

The previous observation is also explained by the following: Recall that *Martin's Maximum* states that whenever $\mathbb{P}$ is a poset which preserves stationary subsets of ω_1 and $\mathcal{D}$ is an ω_1-sized collection of open dense subsets of $\mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

Lemma 3.4.11. *Assume Martin's Maximum. Then any $<\omega_1$-distributive forcing which preserves stationary subsets of ω_1 is strongly $<\omega_1$-distributive.*

Proof. Let $(D_\alpha)_{\alpha<\omega_1}$ be a sequence of open dense subsets of $\mathbb{P}$ and $p \in \mathbb{P}$. Assume for simplicity that $p = 1_\mathbb{P}$ (otherwise work with $\mathbb{P} \upharpoonright p$). Note that we cannot directly apply MM to $\{D_\alpha \mid \alpha < \omega_1\}$ since the resulting sequence might not be descending. However, by the $<\omega_1$-distributivity of $\mathbb{P}$ we can let $(A_\alpha)_{\alpha<\omega_1}$ be a descending sequence of maximal antichains of $\mathbb{P}$ (with regards to refinement) such that $A_\alpha \subseteq D_\alpha$. For any $\alpha < \omega_1$, let E_α be the downward closure of A_α and let $G \subseteq \mathbb{P}$ be a filter intersecting every E_α. For $\alpha \in \omega_1$, let p_α be the unique element of $A_\alpha \cap G$ (it exists since G is upwards closed and is unique because A_α is an antichain and all conditions in G are compatible). Then $(p_\alpha)_{\alpha<\omega_1}$ is descending because for any $\beta < \alpha$, p_α is below some element $p'_\beta \in A_\beta$ which is necessarily equal to p_β because p'_β and p_β are in G and thus compatible. $\square$

Note that the above is also the maximum we can hope for since there always exist $<\omega_1$-distributive forcing notions which destroy stationary subsets of ω_1 and are thus not strongly $<\omega_1$-distributive.

The statement "any stationary set preserving $<\omega_1$-distributive forcing notion is strongly $<\omega_1$-distributive" captures a number of consequences of Martin's Maximum. E.g. it implies that there does not exist a Suslin tree and that any stationary subset of $E^{\omega_2}_\omega := \{\alpha \in \omega_2 \mid \operatorname{cf}(\alpha) = \omega\}$ contains a closed subset with ordertype ω_1.

CHAPTER 4

Orders on Products

Some of the difficulty in working with Mitchell Forcing comes from the fact that it does not quite fit into the general framework of either product forcing or iterated forcing. Therefore, to help in the later sections, this section will introduce a general way of working with arbitrary orders on products of sets. This material is due to the author (from [Jak23] and [Jak24b]).

4.1 Working with Arbitrary Orders on Products of Sets

Definition 4.1.1. Let $\mathbb{P}$ and $\mathbb{Q}$ be nonempty sets and R a partial order on $\mathbb{P} \times \mathbb{Q}$. We will only consider orderings where for all p, p'

$$\exists q_0, q_1((p, q_0)R(p', q_1)) \longrightarrow \forall q((p, q)R(p', q))$$

If we want to reference this property, we will say that $(\mathbb{P} \times \mathbb{Q}, R)$ is *based*. We define the following partial orders:

1. The *base ordering* $b(R)$ is an ordering on $\mathbb{P}$ given by $p(b(R))p'$ if there are $q_0, q_1 \in \mathbb{Q}$ such that $(p, q_0)R(p', q_1)$.
2. The *term ordering* $t(R)$ is an ordering on $\mathbb{P}\times\mathbb{Q}$ given by $(p, q)(t(R))(p', q')$ if $(p, q)R(p', q')$ and $p = p'$.
3. For $p \in \mathbb{P}$, the *section ordering* $s(R, p)$ is an ordering on $\mathbb{Q}$ given by $q(s(R, p))q'$ if $(p, q)R(p, q')$.

We also fix the following properties:

1. $(\mathbb{P} \times \mathbb{Q}, R)$ has the *projection property* if whenever $(p', q')R(p, q)$, there is $q'' \in \mathbb{Q}$ such that $(p, q'')R(p, q)$ and $(p', q'')R(p', q')R(p', q'')$.
2. $(\mathbb{P} \times \mathbb{Q}, R)$ has the *refinement property* if $p'(b(R))p$ implies that $s(R, p')$ refines $s(R, p)$, i.e. whenever $(p, q')R(p, q)$ and $p'(b(R))p$, also $(p', q')R(p', q)$.
3. $(\mathbb{P}\times\mathbb{Q}, R)$ has the *mixing property* if whenever $(p, q_0), (p, q_1)R(p, q)$, there are $p_0, p_1 \in \mathbb{P}$ and $q' \in \mathbb{Q}$ with $(p, q')R(p, q)$ such that $(p_i, q')R(p, q_i)$ for $i = 0, 1$.

We say that $(\mathbb{P}\times\mathbb{Q}, R)$ is *iteration-like* if $(\mathbb{P}\times\mathbb{Q}, R)$ is based and has the projection property, the refinement property and the mixing property.

The projection and refinement property hold in almost all cases, and always for iterations and products. They are necessary for most of the relevant techniques. The mixing property roughly states that we can mix elements of $\mathbb{Q}$ modulo $\mathbb{P}$ and holds e.g. in iterations $\mathbb{P} * \dot{\mathbb{Q}}$ if $\mathbb{P}$ is atomless.

We note that as long as we do not "recompute" our orders in extensions of the universe but work with them "literally", all of the properties above are Δ_0 and thus absolute.

We also note that the term ordering is the disjoint union of the section orderings, so if the term ordering is $< \delta$-closed (strategically closed, strongly distributive etc.) for some δ, so are all the section orderings (and vice versa).

Remark 4.1.2. $b(R)$ actually is a partial order if $(\mathbb{P} \times \mathbb{Q}, R)$ is based since $p'(b(R))p$ if and only if for all $q \in \mathbb{Q}$, $(p', q)R(p, q)$.

Lemma 4.1.3. *If* $(\mathbb{P} \times \mathbb{Q}, R)$ *is based and has the projection and refinement property, the identity is a projection from* $(\mathbb{P}, b(R)) \times (\mathbb{Q}, s(R, 1_{\mathbb{P}}))$ *onto* $(\mathbb{P} \times \mathbb{Q}, R)$.

Proof. Denote by R_π the order on $(\mathbb{P}, b(R)) \times (\mathbb{Q}, s(R, 1_{\mathbb{P}}))$. If $(p', q')R_\pi(p, q)$, $p'(b(R))p$ and $(1, q')R(1, q)$. By the refinement property, $(p, q')R(p, q)$. By basedness $(p', q')R(p, q')$. In summary,

$$(p', q')R(p, q')R(p, q)$$

Assume $(p', q')R(p, q)$. This implies $p'(b(R))p$. Furthermore, by basedness $(p, q)R(1_{\mathbb{P}}, q)$. By the projection property there is q'' such that $(1_{\mathbb{P}}, q'')R(1_{\mathbb{P}}, q)$ and $(p', q'')R(p', q')R(p', q'')$. So $(p', q'')R_\pi(p, q)$ and $(p', q'')R(p', q')$. $\square$

We will now generalise both the product and the factor lemma, showing how we can view forcing with $(\mathbb{P} \times \mathbb{Q}, R)$ as successive forcing. To this end, fix a forcing $(\mathbb{P} \times \mathbb{Q}, R)$ that is based and has the projection as well as the refinement property.

Lemma 4.1.4. *There exists a projection from* $(\mathbb{P} \times \mathbb{Q}, R)$ *onto* $(\mathbb{P}, b(R))$.

Proof. The projection is simply given by $\pi(p, q) = p$. Basedness implies that π is actually a projection: If $(p', q')R(p, q)$, then $\pi(p', q') = p'(b(R))p = \pi(p, q)$ by the definition of $b(R)$. If $p'(b(R))p = \pi(p, q)$, then by basedness $(p', q)R(p, q)$ and $\pi(p', q) = p'(b(R))p'$. $\square$

For the rest of the section, $\mathbb{P}$ refers to $(\mathbb{P}, b(R))$. By the definitions, whenever $G \subseteq \mathbb{P}$ is generic, $(\mathbb{P} \times \mathbb{Q})/G = \{(p, q) \in \mathbb{P} \times \mathbb{Q} \mid p \in G\} = G \times \mathbb{Q}$. We will now show that $G \times \mathbb{Q}$ with the ordering induced by R is forcing equivalent to a particular ordering on $\mathbb{Q}$.

Definition 4.1.5. Let G be $\mathbb{P}$-generic. In $V[G]$, define the *generic ordering* $g(R, G)$ on $\mathbb{Q}$ by $q(g(R, G))q'$ if for some $p \in G$, $(p, q)R(p, q')$

Necessarily we need to show the following:

Remark 4.1.6. $g(R, G)$ is a partial order on $\mathbb{Q}$: Reflexivity is clear. For transitivity, assume that $q_0(g(R, G))q_1(g(R, G))q_2$, i.e. for $p, p' \in G$, $(p, q_0)R(p, q_1)$ and $(p', q_1)R(p', q_2)$. Let $p''(b(R))p, p'$ be in G. Then by the refinement property,

$$(p'', q_0)R(p'', q_1)R(p'', q_2)$$

hence $q_0(g(R, G))q_2$.

Lastly, we show that the generic ordering actually works as intended:

Lemma 4.1.7. *Let G be $\mathbb{P}$-generic. In $V[G]$, the posets $(G\times\mathbb{Q}, R \upharpoonright (G\times\mathbb{Q}))$ and $(\mathbb{Q}, g(R, G))$ are forcing equivalent.*

Proof. Let $\pi\colon (G \times \mathbb{Q}, R \upharpoonright (G \times \mathbb{Q})) \to (\mathbb{Q}, g(R, G))$ be given by $\pi(p, q) = q$. We will verify that π is a trivial projection.

$\pi((1_{\mathbb{P}}, 1_{\mathbb{Q}})) = 1_{\mathbb{Q}}$. Let $(p', q')R(p, q)$. By the projection property, there is q'' with $(p, q'')R(p, q)$ and $(p', q'')R(p', q')R(p', q'')$. Thus p witnesses $q''(g(R, G))q$ and p' witnesses $q'(g(R, G))q''$. By transitivity, $q'(g(R, G))q$.

Assume $(p, q) \in G \times \mathbb{Q}$ and $q'(g(R, G))q$. Let $p' \in G$ witness this, i.e. $(p', q')R(p', q)$. Let $p''(b(R))p', p$. Thus, by the refinement property, $(p'', q')R(p'', q)R(p, q)$ and (p'', q') is as required.

Lastly, if $\pi(p, q) = q = \pi(p', q)$, then let $G \ni p''(b(R))p, p'$. Thus, $(p'', q)R(p', q), (p, q)$. $\square$

In summary, forcing with $(\mathbb{P} \times \mathbb{Q}, R)$ can be regarded as forcing first with $(\mathbb{P}, b(R))$ and then with $(\mathbb{Q}, g(R, G))$, where G is $\mathbb{P}$-generic.

The following results are especially important for Mitchell forcing: In many cases, the term ordering on an iteration $\mathbb{P}*\dot{\mathbb{Q}}$ is $<\kappa$-closed because $\mathbb{P}$ forces $\dot{\mathbb{Q}}$ to be $<\check{\kappa}$-closed. In these cases (viewing this iteration as an ordering on a product), one obtains that $(\mathbb{Q}, g(R, G))$ is forcing equivalent to a $<\kappa$-closed forcing and thus has nice regularity properties. However, as we will later see, there are cases where the term ordering on $\mathbb{P} \times \mathbb{Q}$ is $<\kappa$-closed but $(\mathbb{Q}, g(R, G))$ fails to be. We will now see that, if $\mathbb{P}$ has a good enough chain condition, $(\mathbb{Q}, g(R, G))$ is at least strongly $<\kappa$-distributive.

Lemma 4.1.8. *Assume that the base ordering $(\mathbb{P}, b(R))$ is κ-c.c. and the term ordering $(\mathbb{P} \times \mathbb{Q}, t(R))$ is strongly $<\kappa$-distributive. If $G \subseteq \mathbb{P}$ is generic, in $V[G]$, the ordering $(\mathbb{Q}, g(R, G))$ is strongly $<\kappa$-distributive.*

The proof consists of two simple lemmas.

Lemma 4.1.9. *If $\pi\colon \mathbb{R} \to \mathbb{S}$ is a projection and $\mathbb{R}$ is strongly $<\kappa$-distributive, so is $\mathbb{S}$.*

Proof. If $D \subseteq \mathbb{S}$ is open dense, so is $\pi^{-1}[D]$. Given a sequence $(D_\alpha)_{\alpha<\kappa}$ of open dense subsets of $\mathbb{S}$, find a thread through $(\pi^{-1}[D_\alpha])_{\alpha<\kappa}$ and apply π to it. $\square$

Lemma 4.1.10. *Let G be $\mathbb{P}$-generic. In $V[G]$, there exists a projection from $(\mathbb{Q}, s(R, 1_{\mathbb{P}}))$ onto $(\mathbb{Q}, g(R, G))$.*

Proof. The projection once again is just the identity. $g(R,G)$ is of course finer than $s(R,1_{\mathbb{P}})$, since $1_{\mathbb{P}} \in G$. If $q'(g(R,G))q$, then $(p,q')R(p,q)R(1,q)$ for some $p \in G$, so there exists q'' such that $(1,q'')R(1,q)$ and $(p,q'')R(p,q')R(p,q'')$. Hence $q''(s(R,1_{\mathbb{P}}))q$ and $q''(g(R,G))q'$, witnessed by p. $\square$

Proof of Lemma 4.1.8. In $V[G]$, the ordering $(\mathbb{Q}, s(R,1_{\mathbb{P}}))$ is strongly $<\kappa$-distributive by Lemma 3.3.2. By the previous two lemmas this implies that $(\mathbb{Q}, g(R,G))$ is strongly $<\kappa$-distributive as well. $\square$

We have seen that we can project onto $(\mathbb{P}\times\mathbb{Q}, R)$ from a particular product forcing. In this last lemma we show that the quotient is well-behaved:

Lemma 4.1.11. *Let $(\mathbb{P}\times\mathbb{Q}, R)$ be based and have the projection and refinement property. Assume the base ordering is μ-c.c. and the term ordering is μ-strategically closed. Then the quotient of $(\mathbb{P}, b(R)) \times (\mathbb{Q}, s(R,1_{\mathbb{P}}))$ by $(\mathbb{P}\times\mathbb{Q}, R)$ is $<\mu$-distributive.*

Proof. Let $G\times H$ be $(\mathbb{P}, b(R)) \times (\mathbb{Q}, s(R,1_{\mathbb{P}}))$-generic. Let I be the $(\mathbb{P}\times\mathbb{Q}, R)$-generic filter induced by $G\times H$. Let $f \in V[G\times H]$ be a $<\mu$-sequence of ordinals. Since $(\mathbb{Q}, s(R,1_{\mathbb{P}}))$ is μ-distributive in $V[G]$, $f \in V[G]$. Since the projections

$$(\mathbb{P}, b(R)) \times (\mathbb{Q}, s(R,1_{\mathbb{P}})) \to (\mathbb{P}\times\mathbb{Q}, R) \to (\mathbb{P}, b(R))$$

commute, G is equal to the $(\mathbb{P}, b(R))$-generic filter induced by I. Consequently, $V[G] \subseteq V[I]$ which shows the statement. $\square$

4.2 The Approximation Property

One of the most important lemmas in Mitchell's original work proving the consistency of the tree property at a successor cardinal states that forcing with quotients of his forcing notion over initial segments does not add "fresh sets". Specifically, Lemma 3.8 in [Mit72] states (in slightly different notation): "Suppose $\mathrm{cf}(\gamma)^V > \Theta$, $t\colon \gamma \to V$, $t \in V[G]$ and $t \upharpoonright \alpha \in V[G \upharpoonright \nu]$ for every $\alpha < \gamma$. Then $t \in V[G \upharpoonright \nu]$". Later, this property was explicitely defined by Hamkins in [Ham03] as follows:

Definition 4.2.1. A pair (V,W) of models of set theory has the $<\delta$-approximation property if there is no $A \in W \setminus V$ such that $A \cap a \in V$ for any $a \in [V]^{<\delta} \cap V$. A poset $\mathbb{P}$ has the $<\delta$-approximation property if $(V, V[G])$ has it for every $\mathbb{P}$-generic filter G.

Much work has been done in trying to obtain sufficient conditions that imply a certain forcing order has the $<\delta$-approximation property. Hamkins showed in [Ham99] that iterations of the form $\mathbb{P} * \dot{\mathbb{Q}}$ where $\mathbb{P}$ is atomless, $|\mathbb{P}| \leq \beta$ and $\dot{\mathbb{Q}}$ is forced to be $<\beta^+$-strategically closed have the $<\beta^+$-approximation property. In [Mit06], Mitchell weakened the requirement $|\mathbb{P}| \leq \beta$ to requiring simply that $\mathbb{P}$ be *strongly proper on a stationary set* and worked with generalizations of iterated forcing (which are very close to our work on orders on products). Unger showed in [Ung15] that iterations of the form $\mathbb{P} * \dot{\mathbb{Q}}$ where $\mathbb{P}\times\mathbb{P}$ is δ-c.c. and

$\dot{\mathbb{Q}}$ is forced to be $<\delta$-closed have the $<\delta$-approximation property. Additionally, Krueger showed in [Kru08a] that his *mixed support iterations* also have the approximation property (with the parameter depending on the cardinals used).

In this section, we will produce a further generalization of Unger's result (albeit with a very similar proof). In one direction, we work with arbitrary orders on products, similar to Mitchell in [Mit06], which is helpful when trying to apply the result to variants of Mitchell Forcing. In a more substantial improvement, we only require that the term ordering be strongly distributive instead of strategically closed which has the advantage that this property is often preserved when taking forcing extensions by small posets (unlike strategic closure which is almost lost). This will be important in later chapters where we want a forcing order $\mathbb{M}(\kappa \setminus \nu)$ to retain the $<\mu$-approximation property even after forcing with $\mathrm{Add}(\tau)$ and could have applications when trying to construct models where ISP is indestructible under further forcing with posets which have sufficiently small antichains.

Theorem 4.2.2. *Let $(\mathbb{P}\times\mathbb{Q},R)$ be an iteration-like partial order and δ a cardinal. Assume $(\mathbb{P},b(R))^2$ is δ-c.c. and $(\mathbb{P}\times\mathbb{Q},t(R))$ is strongly $<\delta$-distributive. Then $(\mathbb{P}\times\mathbb{Q},R)$ has the $<\delta$-approximation property.*

We begin with a helping lemma:

Lemma 4.2.3. *Let $(\mathbb{P}\times\mathbb{Q},R)$ be an iteration-like partial order. If (p,q) forces $\dot{x}\in V$ but for any $y\in V$, $(p,q)\not\Vdash \dot{x}=\check{y}$, there are $q''\in\mathbb{Q}$, $p_0,p_1 b(R)p$ and $y_0\neq y_1$ such that $(p,q'')R(p,q)$ and for $i\in 2$, $(p_i,q'')\Vdash\dot{x}=\check{y}_i$.*

Proof. For better readability, we prove the result in a series of statements, showing where we apply which property. We let PP stand for the projection property, RP for the refinement property and MP for the mixing property. Let BS stand for basedness. We check two cases:

$$\begin{aligned}
\text{Case 1:}\quad & \exists q_0,y_0(((p,q_0)R(p,q))\wedge((p,q_0)\Vdash\dot{x}=\check{y}_0))\\
& \exists(p',q'''),y_1\neq y_0(((p',q''')R(p,q))\wedge((p',q''')\Vdash\dot{x}=\check{y}_1)\\
(PP)\quad & \exists q_1((p,q_1)R(p,q)\wedge((p',q_1)R(p',q''')))\\
& (p',q_1)\Vdash\dot{x}=\check{y}_1\\
& p'(b(R))p\\
(BS)\quad & ((p',q_0)R(p,q_0))\\
& (p',q_0)\Vdash\dot{x}=\check{y}_0\\
(RP)\quad & ((p',q_0)R(p',q))\wedge((p',q_1)R(p',q))\\
(MP)\quad & \exists p_0,p_1,q'(((p',q')R(p',q))\wedge((p_0,q')R(p',q_0))\wedge((p_1,q')R(p',q_1)))\\
& p_0,p_1(b(R))p'\\
(BS)\quad & (p',q')R(p',q)R(p,q)\\
(PP)\quad & \exists q''((p,q'')R(p,q))\wedge((p',q'')R(p',q')R(p',q''))\\
(RP)\quad & ((p_0,q'')R(p_0,q')R(p',q_0))\wedge((p_1,q'')R(p_1,q')R(p',q_1))\\
& ((p_0,q'')\Vdash\dot{x}=\check{y}_0)\wedge((p_1,q'')\Vdash\dot{x}=\check{y}_1)
\end{aligned}$$

$$
\begin{array}{rl}
\text{Case 2:} & \forall q_0, y_0(((p, q_0)R(p, q)) \to ((p, q_0) \nVdash \dot{x} = \check{y}_0)) \\
& \exists(p_0, q'), y_0(((p_0, q')R(p, q)) \wedge ((p_0, q') \Vdash \dot{x} = \check{y}_0)) \\
(PP) & \exists q'''(((p, q''')R(p, q)) \wedge ((p_0, q''')R(p_0, q')R(p_0, q'''))) \\
& ((p, q''') \nVdash \dot{x} = \check{y}_0) \\
& \exists(p_1, q''''), y_1 \neq y_0(((p_1, q'''')R(p, q''')) \wedge ((p_1, q'''') \Vdash \dot{x} = \check{y}_1)) \\
(PP) & \exists q''((p, q'')R(p, q''') \wedge ((p_1, q'')R(p_1, q'''')R(p_1, q''))) \\
& ((p_1, q'') \Vdash \dot{x} = \check{y}_1) \\
& p_0, p_1(b(R))p \\
& (p, q'')R(p, q''')R(p, q) \\
(RP) & (p_0, q'')R(p_0, q''')R(p_0, q') \\
& ((p_0, q'') \Vdash \dot{x} = \check{y}_0)
\end{array}
$$

□

Now we can finish the proof of Theorem 4.2.2.

Proof of Theorem 4.2.2. Let $\dot{f}$ be a $\mathbb{P} \times \mathbb{Q}$-name for a function such that some (p, q) forces $\dot{f} \notin V$ and $\dot{f} \restriction \check{u} \in V$ for every $u \in [V]^{<\delta} \cap V$. We will construct a winning strategy for INC in the completeness game of length δ played on $(\mathbb{Q}, s(R, p)) \restriction q$ (this suffices because INC can arbitrarily decide the first played condition). In any run $(q_\gamma)_{\gamma \in \delta}$ of the game, we will construct $(p^0_\gamma, p^1_\gamma, y_\gamma)_{\gamma \in \mathrm{Odd}}$ such that

1. $y_\gamma \in [V]^{<\delta} \cap V$ and the sequence $(y_\gamma)_{\gamma \in \mathrm{Odd}}$ is $\subseteq$-increasing
2. $p^0_\gamma, p^1_\gamma b(R)p$
3. (p^0_γ, q_γ) and (p^1_γ, q_γ) decide $\dot{f} \restriction \check{y}_\alpha$ equally for any odd $\alpha < \gamma$, but differently for $\alpha = \gamma$

Assume the game has been played until some even ordinal $\gamma < \delta$. Let $y'_{\gamma+1} := \bigcup_{\alpha \in \gamma \cap \mathrm{Odd}} y_\alpha$, which has size $< \delta$. $\dot{f} \restriction y'_{\gamma+1}$ is forced to be in V, so we can find $(p'_{\gamma+1}, q'_{\gamma+1})R(p, q_\gamma)$ which decides $\dot{f} \restriction y'_{\gamma+1}$. By the projection property, we can find $q''_{\gamma+1}$ such that $(p, q''_{\gamma+1})R(p, q_\gamma)$ and $(p'_{\gamma+1}, q''_{\gamma+1})R(p'_{\gamma+1}, q'_{\gamma+1})$, so $(p'_{\gamma+1}, q''_{\gamma+1})$ also decides $\dot{f} \restriction \check{y}'_{\gamma+1}$.

Because $\dot{f}$ is forced to be outside of V, there is β such that $(p'_{\gamma+1}, q''_{\gamma+1})$ does not decide $\dot{f}(\check{\beta})$. Define $y_{\gamma+1} := y'_{\gamma+1} \cup \{\beta\}$. Then $(p'_{\gamma+1}, q''_{\gamma+1})$ does not decide $\dot{f} \restriction \check{y}_{\gamma+1}$, so we find $q'''_{\gamma+1}$ and $p^0_{\gamma+1}, p^1_{\gamma+1}$ such that $(p'_{\gamma+1}, q'''_{\gamma+1})R(p'_{\gamma+1}, q''_{\gamma+1})$ and $(p^0_{\gamma+1}, q'''_{\gamma+1})$ and $(p^1_{\gamma+1}, q'''_{\gamma+1})$ decide $\dot{f} \restriction \check{y}_{\gamma+1}$ differently. Lastly, use the projection property to obtain $q_{\gamma+1}$ such that $(p, q_{\gamma+1})R(p, q''_{\gamma+1})$ and $(p'_{\gamma+1}, q_{\gamma+1})R(p'_{\gamma+1}, q'''_{\gamma+1})$. It follows that these objects are as required.

Lastly, assume this strategy does not win, i.e. there is a game of length δ. In this case, we claim that $\{(p^0_\gamma, p^1_\gamma) \mid \gamma \in \mathrm{Odd}\}$ is an antichain in $(\mathbb{P}, b(R))^2$, obtaining a contradiction. Assume $(p^0, p^1)(b(R)^2)(p^0_\gamma, p^1_\gamma), (p^0_{\gamma'}, p^1_{\gamma'})$ with $\gamma' < \gamma$. Because $p^0(b(R))p^0_\gamma$ and $p^1(b(R))p^1_\gamma$, $(p^0, q_\gamma)R(p^0_\gamma, q_\gamma)$ and $(p^1, q_\gamma)R(p^1_\gamma, q_\gamma)$ decide $\dot{f} \restriction \check{y}_{\gamma'}$ equally, but because $p^0(b(R))p^0_{\gamma'}$ and

$p^1(b(R))p^1_{\gamma'}$, $(p^0, q_\gamma)R(p^0_{\gamma'}, q_\gamma)R(p^0_{\gamma'}, q_{\gamma'})$ (using the refinement property) and $(p^1, q_\gamma)R(p^1_{\gamma'}, q_\gamma)$ $R(p^1_{\gamma'}, q_{\gamma'})$ decide $\dot{f} \restriction \check{y}_{\gamma'}$ differently, a contradiction. $\square$

We obtain an interesting statement regarding strongly distributive forcings (which contrasts the fact that consistently there can exist a $<\omega_1$-distributive, c.c.c. poset).

Corollary 4.2.4. *Let $\mathbb{P}$ be a nontrivial poset and δ a cardinal. Then one of the following holds:*

1. *$\mathbb{P}$ is not δ-c.c.*
2. *$\mathbb{P}$ is not strongly $<\delta$-distributive.*

Proof. Assume to the contrary that none of the above holds. As $\mathbb{P}$ is nontrivial, it adds some set x of ordinals. Because $\mathbb{P}$ is in particular $<\delta$-distributive, $x \cap y \in V$ for every $y \in [x]^{<\delta} \cap V$. Thus $\mathbb{P}$ does not have the $<\delta$-approximation property. On the other hand, $\mathbb{P}$ preserves the δ-c.c. of itself by virtue of being strongly $<\delta$-distributive (by Lemma 3.3.2), so $\mathbb{P} \times \mathbb{P}$ has the δ-c.c. and $\mathbb{P}$ has the $<\delta$-approximation property (by viewing $\mathbb{P}$ as an order on the product $\mathbb{P} \times \{0\}$), a contradiction. $\square$

4.3 Examples of Strongly Distributive Forcings

With the results from the previous section, we can give some examples of strongly distributive forcings to show the following:

1. There is no provable relationship between $<\lambda^+$-strategic closure and strong $<\lambda^+$-distributivity. There can consistently be a forcing which is $<\lambda^+$-strategically closed but not strongly $<\lambda^+$-distributive and there can consistently be a forcing which is strongly $<\lambda^+$-distributive but not $<\lambda^+$-strategically closed.
2. Strong $<\lambda^+$-distributivity is not downwards absolute.
3. Strongly $<\kappa$-distributive forcings can destroy the stationarity of subsets of $[\lambda]^{<\kappa}$. In particular, strongly $<\omega_1$-distributive forcings are not necessarily proper.
4. There can be forcings $\mathbb{P}$ and $\mathbb{Q}$ such that $\mathbb{P} \times \mathbb{Q}$ (with the product ordering) is strongly $<\kappa$-distributive but $\mathbb{Q}$ is no longer strongly $<\kappa$-distributive after forcing with $\mathbb{P}$. Furthermore, there can be forcings $\mathbb{P}$ and $\mathbb{Q}$ such that $\mathbb{P}$ is $<\kappa$-closed and $\mathbb{Q}$ is strongly $<\kappa$-distributive but $\mathbb{Q}$ is no longer strongly $<\kappa$-distributive after forcing with $\mathbb{P}$.

Example 4.3.1. Let $\mathbb{P}$ be the forcing to add a $\square_\lambda$-sequence. Conditions are functions p such that

1. $\mathrm{dom}(p) = \{\beta \leq \alpha \mid \lim(\beta)\}$ for some limit ordinal $\alpha \in \lambda^+$.

2. For all $\alpha \in \mathrm{dom}(p)$, $p(\alpha)$ is club in α of ordertype $\leq \lambda$.

3. Whenever β is a limit point of $p(\alpha)$, $p(\beta) = p(\alpha) \cap \beta$.

ordered by extension.

Classically, this forcing is $< \lambda^+$-strategically closed. However, if $\Box_\lambda$ fails, the forcing is not strongly $< \lambda^+$-distributive: Let $(D_\alpha)_{\alpha<\lambda^+}$ be a sequence such that $D_\alpha := \{p \in \mathbb{P} \mid \alpha \in \mathrm{dom}(p)\}$ if α is a limit ordinal and $D_\alpha := \mathbb{P}$ otherwise. If $(p_\alpha)_{\alpha<\lambda^+}$ is a thread through $(D_\alpha)_{\alpha<\lambda^+}$, $\bigcup_{\alpha<\lambda^+} p_\alpha$ is a $\Box_\lambda$-sequence, a contradiction.

The above example also shows (2): After forcing with $\mathbb{P}$, $\Box_\lambda$ holds, so by [IY02, Theorem 3.3], every $< \lambda^+$-strategically closed poset (and in particular, $\mathbb{P}$) is λ^+-strategically closed and thus strongly $< \lambda^+$-distributive.

Example 4.3.2. Let $\mathbb{P}$ be $\mathrm{Add}(\omega_1)$. Let G be an $\mathrm{Add}(\omega)$-generic filter. In $V[G]$, $\mathbb{P}$ is still strongly $< \omega_1$-distributive. Assume $\mathbb{P}$ is $< \omega_1$-strategically closed in $V[G]$, i.e. $\mathrm{Add}(\omega)$ forces that $\check{\mathbb{P}}$ is $< \omega_1$-strategically closed. This implies that the term ordering on $\mathrm{Add}(\omega) * \check{\mathbb{P}}$ is $< \omega_1$-strategically closed. By a classical fact of Jech, in this case (for ω_1), the term ordering on $\mathrm{Add}(\omega) * \check{\mathbb{P}}$ is actually ω_1-strategically closed and thus strongly $< \omega_1$-distributive. By Lemma 4.2.2 the poset $\mathrm{Add}(\omega) * \check{\mathbb{P}}$ has the $< \omega_1$-approximation property. This is obviously not the case as it is equivalent to $\mathrm{Add}(\omega) \times \mathrm{Add}(\omega_1)$.

Example 4.3.3. Assume $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$. Let $\mathbb{P}$ be the forcing to collapse ω_2 by adding a cofinal, continuous sequence of length ω_1 to $[\omega_2]^{<\omega_1}$. Conditions are functions p such that

1. $\mathrm{dom}(p)$ is a successor ordinal.

2. for every $\alpha \in \mathrm{dom}(p)$, $p(\alpha) \in [\omega_2]^{<\omega_1}$. For every limit $\delta \in \mathrm{dom}(p)$, $p(\delta) = \bigcup_{\alpha<\delta} p(\alpha)$

ordered by end-extension. This forcing is $< \omega_1$-closed.

Now let G be $\mathrm{Add}(\omega) * \dot{\mathrm{Coll}}(\check{\omega}_1, \check{\omega}_2)$-generic and let H be the induced $\mathrm{Add}(\omega)$-generic filter. In $V[G]$, ω_2^V has size ω_1. Furthermore, in this extension, $\mathbb{P}$ still adds a cofinal and continuous sequence through $([\omega_2]^{<\omega_1})^V$. The range of this sequence is forced to be a club in $[\omega_2^V]^{<\omega_1}$: It is clearly closed and it is unbounded by the countable covering property of $\mathrm{Add}(\omega) * \dot{\mathrm{Coll}}(\check{\omega}_1, \check{\omega}_2)$. By the later discussion, $[\omega_2^V]^{<\omega_1} \setminus V$ is still stationary inside $[\omega_2^V]^{<\omega_1}$ in $V[G]$ (using the internal approachability), so $\mathbb{P}$ (in $V[G]$) destroys a stationary subset of $[\omega_2^V]^{<\omega_1}$. Since ω_2^V is of size ω_1 in $V[G]$, $\mathbb{P}$ cannot be strongly $< \omega_1$-distributive in this model. However, $\mathbb{P}$ is strongly $< \omega_1$-distributive in $V[H]$, so its strong distributivity is destroyed by the $< \omega_1$-closed forcing $\mathrm{Coll}(\omega_1, \omega_2)$. Furthermore, in $V[H]$, $\mathrm{Coll}(\omega_1, \omega_2) \times \mathbb{P}$ is (forcing equivalent to) a strongly $< \omega_1$-distributive forcing, because the term ordering on $\mathrm{Add}(\omega) \times (\dot{\mathrm{Coll}}(\check{\omega}_1, \check{\omega}_2) \times \mathbb{P})$ is $< \omega_1$-closed.

CHAPTER 5

Embedding Characterisations of Small Large Cardinals

Many large cardinal properties are witnessed by the existence of elementary embeddings from the universe into a transitive class. E.g. a cardinal κ is measurable if and only if there exists an elementary embedding $j\colon V \to M$ such that $j \restriction \kappa = id_\kappa$, but $j(\kappa) > \kappa$. However, this directly implies that cardinals of smaller consistency strength (such as Mahlo or weakly compact cardinals) cannot be assumed to be the critical point of an elementary embedding from the universe. Despite that, for many of these cardinals it has been observed that we can state their large cardinal property in the form of elementary submodels of $H(\Theta)$ (or equivalently, elementary embeddings from small transitive structures into $H(\Theta)$), allowing us to use well-known techniques about lifting such embeddings.

Most of the results in this chapter are either part of the folklore or adapted from [HLN19]. The results about Laver functions for Mahlo and λ-ineffable cardinals are unpublished and due to the author.

5.1 General Results

We state and prove here some general results which will help in the coming sections, starting with the Mostowski-Collapse.

Lemma 5.1.1. *Let $(M, \in)$ be an extensional structure (i.e. if $x, y \in M$ and $x \neq y$ there is $z \in M$ such that $z \in x \Delta y$). There is a unique pair (N, π) such that N is transitive and $\pi\colon (M, \in) \to (N, \in)$ is an isomorphism.*

Proof. We define π by $\in$-induction, letting

$$\pi(x) := \{\pi(y) \mid y \in x \cap M\}$$

We leave it to the reader to show that N is transitive (π is an isomorphism because of the extensionality of M).

To show the uniqueness of (N, π) it suffices to show that whenever N and N' are both transitive and isomorphic via some σ, $\sigma = id$ (and so $N = N'$). This again follows by $\in$-induction: Assume $y \in N$ and $\sigma(x) = x$ for every $x \in y$. This directly implies $y \subseteq \sigma(y)$ (if $x \in y$, $x = \sigma(x) \in \sigma(y)$). If $\sigma(x) \in \sigma(y)$ (σ is surjective), $x \in y$, so $\sigma(x) = x$ and thus $\sigma(x) \in y$. This shows $\sigma(y) = y$. $\square$

We fix a definition corresponding to the previous Lemma:

Definition 5.1.2. Let $(M, \in)$ be an extensional structure. The *Mostowski-Collapse of* M is (N_M, π_M), where N_M is transitive and $\pi_M : M \to N_M$ an isomorphism.

We will later be concerned with showing that models are closed under certain sequences. We want to show that it suffices to consider sequences of ordinals. This is especially helpful when forcing because forcing cannot add ordinals.

We need a helping Lemma first:

Lemma 5.1.3. *Assume $M \prec H(\Theta)$ for some large Θ and κ is a cardinal with $M \cap \kappa \in \kappa$. If $x \in M$ has size $< \kappa$, $x \subseteq M$.*

Proof. As $M \prec H(\Theta)$, $|x| \in M$, so $|x| < M \cap \kappa \in \kappa$. Let $f \in M$ be an enumeration of x. By elementarity $f[|x|] = x \subseteq M$ since $|x| \subseteq M$. □

Now we can show the real result:

Lemma 5.1.4. *Let $M \prec H(\Theta)$ for some large Θ. Let (N, π) be its Mostowski-Collapse and $\beta \subseteq M$. The following are equivalent:*

1. ${}^{\beta}M \subseteq M$
2. ${}^{\beta}N \subseteq N$
3. ${}^{\beta}(N \cap \mathrm{On}) \subseteq N$
4. ${}^{\beta}(M \cap \mathrm{On}) \subseteq M$

Proof. We go in a circle.

(1) → (2) Assume $f \in {}^{\beta}N$. Let $g \in {}^{\beta}M$ be such that $\pi(g(\alpha)) = f(\alpha)$. By assumption $g \in M$. Because $\beta \subseteq M$, $g \subseteq M$, so $\pi(g) = \pi[g] = f \in N$.

(2) → (3) Clear.

(3) → (4) Assume $f \in {}^{\beta}(M \cap \mathrm{On})$. Let $g \in {}^{\beta}N$ be such that $g(\alpha) = \pi(f(\alpha))$. Then $g \in {}^{\beta}(N \cap \mathrm{On})$, so $g \in N$. Let $h \in M$ be such that $\pi(h) = g$. Because π is an isomorphism (and $\pi(\beta) = \beta$ by induction), $\mathrm{dom}(h) = \pi(\mathrm{dom}(h)) = \mathrm{dom}(\pi(h)) = \mathrm{dom}(g) = \mathrm{dom}(f)$ and for every $\alpha \in \mathrm{dom}(h)$, $\pi(h(\alpha)) = g(\alpha) = \pi(f(\alpha))$, i.e. $h = f$, so $f \in M$.

(4) → (1) Assume M is closed under β-sequences of ordinals. Let $f \in {}^{\beta}M$. Let g be defined by $g(\alpha) = \mathrm{rk}(f(\alpha))$. Then $g \in {}^{\beta}(M \cap \mathrm{On}) \subseteq M$, so $\alpha := \sup(\mathrm{im}(g)) + 1 \in M$ and in particular $V_\alpha \in M$. Let $\lambda \in M$ be such that there is a bijection $F : V_\alpha \to \lambda$ in M. Then $F \circ f \in {}^{\beta}(M \cap \mathrm{On}) \subseteq M$ which implies $f \in M$, as $F \in M$.

□

5.2 Elementary Submodels Witnessing Large Cardinals

In this section, we give equivalent definitions for Mahlo and λ-ineffable cardinals in terms of elementary submodels. This is related to the existence of embeddings as follows: If M is an extensional structure, we can assign to M a pair (N, π), the Mostowski-Collapse of M, such that N is transitive and $\pi\colon M \to N$ is an isomorphism. If $M \prec H(\Theta)$, M is extensional and thus isomorphic to a transitive structure N via some collapsing map π. The inverse π^{-1} is an isomorphism between N and an elementary submodel of $H(\Theta)$ and thus an elementary embedding from N into $H(\Theta)$.

Theorem 5.2.1 (Folklore)**.** *Let κ be a cardinal. The following are equivalent:*

1. *κ is a Mahlo cardinal.*
2. *For any Θ large enough and any $x \in H(\Theta)$, there is $M \prec H(\Theta)$ with $x \in M$ such that the following holds:*
   (a) *$|M| = M \cap \kappa \in \kappa$ is inaccessible*
   (b) *${}^{<M\cap\kappa}M \subseteq M$*

Proof. Let κ be Mahlo, Θ large and $x \in H(\Theta)$. Using induction on α, we build an $\in$-increasing and continuous sequence $(M_\alpha)_{\alpha<\kappa}$ of elements of $[H(\Theta)]^{<\kappa}$ such that $x \in M_0$ and for every α, $M_\alpha \prec H(\Theta)$, $M_{\alpha+1} \supseteq {}^{<M_\alpha\cap\kappa}M_\alpha$, $M_{\alpha+1} \cap \kappa \geq |M_\alpha|$. This is possible because κ is inaccessible.

By construction, the set of all α such that $M_\alpha \cap \kappa = \alpha$ is club in κ. Because κ is Mahlo, there is $\nu < \kappa$ such that $M_\nu \cap \kappa = \nu$ and ν is inaccessible. Then M_ν is as required: M_ν is the union of sets of size $<\nu$ of length ν, so $|M_\nu| \leq \nu$ (with $\geq$ being clear) and if $f\colon \alpha \to M_\nu$ for some $\alpha < \nu$, $\mathrm{im}(f) \subseteq M_\beta$ for some $\beta < \nu$ by regularity and thus $f \in M_{\max(\alpha,\beta)+1} \subseteq M_\nu$.

Now assume (2) holds. Let $C \subseteq \kappa$ be club. Let $M \prec H(\Theta)$ with $M \cap \kappa \in \kappa$ inaccessible and $C \in M$. Then $M \cap \kappa$ is unbounded in C by elementarity, so $M \cap \kappa \in C$. □

Now we give a relatively new embedding characterisation, this time for λ-ineffable cardinals. A similar characterisation was found by Holy, Lücke and Njegomir in [HLN19].

Theorem 5.2.2. *Let $\kappa < \lambda$ be cardinals. Let f be a (κ, λ)-list. The following are equivalent:*

1. *f has an ineffable branch*
2. *For all sufficiently large cardinals Θ there is $M \prec H(\Theta)$ of size $<\kappa$ with $M \cap \kappa \in \kappa$ such that $\kappa, \lambda, f \in M$ and for some $b \in M$, $b \cap M = f(M \cap \lambda)$.*

Proof. Let b be an ineffable branch for f, i.e.

$$S := \{x \in [\lambda]^{<\kappa} \mid f(x) = b \cap x\}$$

is stationary in $[\lambda]^{<\kappa}$. Using standard techniques we find $M \prec H(\Theta)$ with $\kappa, \lambda, f, b \in M$ such that $M \cap \kappa \in \kappa$ and $M \cap \lambda \in S$. Clearly $b \cap M = f(M \cap \lambda)$.

Assume the submodel property holds. Find $M \prec H(\Theta)$ of size $<\kappa$ such that $\kappa, \lambda, f \in M$ and for some $b \in M$, $b \cap M = f(M \cap \lambda)$. We will show that b is an ineffable branch for f. If b is not an ineffable branch, by elementarity, there is $C \in M$ club in $[\lambda]^{<\kappa}$ such that for every $x \in C$, $f(x) \neq x \cap b$. M contains a function $F\colon [\lambda]^{<\omega} \to [\lambda]^{<\kappa}$ such that every $x \in [\lambda]^{<\kappa}$ closed under F is in C. Because $F \in M \prec H(\Theta)$, $M \cap \lambda$ itself is closed under F, so $M \cap \lambda \in C$ but $f(M \cap \lambda) = M \cap b$, a contradiction. □

We fix a definition corresponding to the previous Lemma:

Definition 5.2.3. Let $\kappa \leq \lambda$ be cardinals and f a (κ, λ)-list. Let Θ be large. $M \in [H(\Theta)]^{<\kappa}$ is a *λ-ineffability witness for κ with respect to f* if $M \prec H(\Theta)$, $M \cap \kappa \in \kappa$, $\{f, \kappa, \lambda\} \subseteq M$ and there is $b \in M$ such that $b \cap M = f(M \cap \lambda)$.

So in particular, we have shown that a (κ, λ)-list has an ineffable branch if and only if there is a λ-ineffability witness for κ with respect to f. This enables us to show $\mathrm{ISP}(\delta, \kappa, \lambda)$ in forcing extensions the following way: Given a name $\dot{f}$ for a $<\delta$-slender (κ, λ)-list, we construct (in the ground model) a related (κ, λ)-list e (using the slenderness of $\dot{f}$) such that, given a λ-ineffability witness M for κ with respect to e in the ground model and a generic filter G, $M[G]$ is a λ-ineffability witness for κ with respect to $\dot{f}^G$.

Later, we will use embeddings with additional regularity properties:

Lemma 5.2.4. *Let $\kappa \leq \lambda = \lambda^{<\kappa}$ be cardinals and assume κ is λ-ineffable. For all sufficiently large cardinals Θ, every $x \in H(\Theta)$ and every (κ, λ)-list f, there is a λ-ineffability witness M for κ with respect to f such that $M \cap \kappa \in \kappa$ is inaccessible, $x \in M$ and $[M \cap \lambda]^{<M \cap \kappa} \subseteq M$.*

Proof. Let $F\colon \lambda \to [\lambda]^{<\kappa}$ be a bijection. We define a modified (κ, λ)-list g: Let $a \in [\lambda]^{<\kappa}$.

1. $g(a) := \emptyset$ if $a \cap \kappa$ is not an ordinal.
2. $g(a)$ is a cofinal subset of $a \cap \kappa$ of minimal ordertype if $a \cap \kappa$ is a singular ordinal.
3. $g(a)$ is an element of $[a]^{<a \cap \kappa} \setminus F[a]$ if $a \cap \kappa$ is a regular cardinal and such an element exists.
4. $g(a) := f(a)$ otherwise.

Let b_g be an ineffable branch for g, witnessed by a stationary set S. The set C of all $a \in [\lambda]^{<\kappa}$ such that $a \cap \kappa$ is a strong limit cardinal (and in particular an ordinal) is easily seen to be club (since κ is inaccessible by assumption). So we can find $M \prec H(\Theta)$ with $x, \kappa, \lambda, f, b_g \in M$ such that $M \cap \lambda \in S \cap C$. So $M \cap \kappa = M \cap \lambda \cap \kappa$ is a strong limit cardinal and M is a λ-ineffability witness for κ with respect to g. We want to exclude every case but (4) for $a := M \cap \lambda$.

We first show:

Claim. *If $y \in M$, $y \subseteq \lambda$ is such that* $\operatorname{otp}(y \cap M) < M \cap \kappa$, $y \cap M \in M$.

Proof. By assumption, there is in M an order-preserving and bijective function $h\colon \delta \to y$, where $\delta = \operatorname{otp}(y)$. Furthermore, $\operatorname{otp}(y \cap M) \in M$ since $M \cap \kappa$ is an ordinal. Ergo the set $h[\operatorname{otp}(y \cap M)]$ is in M by elementarity. We claim $h[\operatorname{otp}(y \cap M)] = y \cap M$. Of course, $\operatorname{otp}(y \cap M)$ is an initial segment of $\operatorname{otp}(y)$, so $h[\operatorname{otp}(y \cap M)]$ is an initial segment of y (because h is order-preserving). Furthermore, $h[\operatorname{otp}(y \cap M)] \subseteq M$, so $h[\operatorname{otp}(y \cap M)]$ is an initial segment of $y \cap M$. Lastly, both sets have the same ordertype, so they are equal. □

1. $a \cap \kappa = M \cap \lambda \cap \kappa = M \cap \kappa$ is an ordinal by assumption.

2. Assume $a \cap \kappa$ is singular. Then $g(a)$ is a cofinal subset of $a \cap \kappa$ of ordertype $< a \cap \kappa$. By our claim, $g(a) = b_g \cap M \in M$ but then M satisfies that $b_g \cap M$ is a cofinal subset of κ of ordertype $< \kappa$, a contradiction.

3. Assume $[a]^{<a\cap\kappa} \setminus F[a]$ is nonempty, so $g(a) \in [a]^{a\cap\kappa} \setminus F[a]$. Again, by our claim, $g(a) = b_g \cap M \in M$. However, by elementarity, $F[a] = [a]^{<a\cap\kappa} \cap M$, a contradiction.

Ergo $g(a) = f(a)$, so M is a λ-ineffability witness for κ with respect to f. We have already shown $[a]^{<a\cap\kappa} \subseteq F[a]$ and $F[a] \subseteq [a]^{<a\cap\kappa}$ holds by elementarity: If $y \in F[a]$, $y \in M$, so $|y| < M \cap \kappa = a \cap \kappa$ and thus $y \subseteq M \cap \lambda = a$. So $F[a] = [a]^{<a\cap\kappa}$. Hence:

$$[M \cap \lambda]^{<M\cap\kappa} = [a]^{<a\cap\kappa} = F[a] \subseteq M$$

□

We will also make use of ineffability witnesses which have a property reminiscent of Π^1_1-indescribability:

Definition 5.2.5. Let $\kappa \leq \lambda$ be cardinals, $A \subseteq \lambda$ and Θ large enough. $M \prec H(\Theta)$ of size $< \kappa$ containing κ, λ and A is Π^1_1-*correct about* λ *with respect to* A if the following holds: Let ϕ be a first-order formula in the language $\mathcal{L} := \{P_0, P_1, R\}$, where the P_i are unary relation symbols and R is a binary relation symbol. Let $\pi\colon M \to N$ be the Mostowski-Collapse of M. Whenever there are $B \subseteq \pi(\lambda)$ and $x_0, \ldots, x_{n-1} \in \pi(\lambda)$ such that

$$(\pi(\lambda), \pi(A), B, \in) \models \phi[x_0, \ldots, x_{n-1}]$$

then there is such a B in N.

We note that by transitivity of N:

$$(\pi(\lambda), \pi(A), B, \in) \models \phi[x_0, \ldots, x_{n-1}] \text{ if and only if } ((\pi(\lambda), \pi(A), B, \in) \models \phi[x_0, \ldots, x_{n-1}])^N$$

Lemma 5.2.6. *Let $\kappa \leq \lambda = \lambda^{<\kappa}$ be cardinals and $A \subseteq \lambda$. There exists a function F and a (κ, λ)-list e such that whenever M is a λ-ineffability witness for κ with respect to e and $F \in M$, M is Π^1_1-correct about λ with respect to A.*

Proof. Fix a bijection $F\colon \lambda \times \lambda^{<\omega} \times \omega \to \lambda$. Let C be the club in $[\lambda]^{<\kappa}$ consisting of those $a \in [\lambda]^{<\kappa}$ such that $F[a \times a^{<\omega} \times \omega] = a$. We define e on C. So let $a \in C$. Let $n \in \omega$ and $(x_0, \ldots, x_{k-1}) \in a^{<\omega}$. Let $\pi_a\colon a \to \operatorname{otp}(a)$ be the unique order-preserving bijection. If there exists $B \subseteq \operatorname{otp}(a)$ such that $(\operatorname{otp}(a), \pi_a[A \cap a], B) \models \phi[\pi(x_0), \ldots, \pi(x_{k-1})]$, where ϕ has Gödel number n, let $B(x_0, \ldots, x_{k-1}, n) := B$, empty otherwise. Define

$$e(a) := F\left[\bigcup_{(x_0,\ldots,x_{k-1},n)\in\omega\times a^{<\omega}} \pi_a^{-1}[B(x_0, \ldots, x_{k-1}, n)] \times \{(x_0, \ldots, x_{k-1}, n)\}\right]$$

Now let M be a λ-ineffability witness for κ with respect to e with $F \in M$. Let $a := M \cap \lambda$ and let (N, π) be the Mostowski-Collapse of M (we note $a \in C$ because $F \in M$ and M is an elementary submodel). We note $\pi_a = \pi \restriction a$. Let ϕ be a first-order formula with Gödel number n and $\pi(x_0), \ldots, \pi(x_{k-1}) \in \pi(\lambda)$. Assume there is $B \subseteq \pi(\lambda)$ such that $(\operatorname{otp}(a), \pi[A \cap a], B) \models \phi[\pi(x_0), \ldots, \pi(x_{k-1})]$. So $B(n, x_0, \ldots, x_{k-1})$ is nonempty. Since M is a λ-ineffability witness for κ with respect to e, there is $b \in M$ such that $b \cap M = e(M \cap \lambda)$. Let d be the set of all $\alpha \in \lambda$ such that $F(\alpha, x_0, \ldots, x_{k-1}, n) \in b$. By elementarity $d \in M$. Lastly,

$$\pi(d) = \pi[d \cap M] = \pi[\pi_a^{-1}[B(x_0, \ldots, x_{k-1}, n)]] = B(x_0, \ldots, x_{k-1}, n) \in N$$

□

By coding two (κ, λ)-lists into a single one, we obtain:

Corollary 5.2.7. *Let $\kappa \leq \lambda = \lambda^{<\kappa}$ be cardinals and assume κ is λ-ineffable. Let $A \subseteq \lambda$, f be a (κ, λ)-list, Θ sufficiently large and $x \in H(\Theta)$. Then there is a λ-ineffability witness $M \prec H(\Theta)$ for κ with respect to f such that M is Π^1_1-correct about λ with respect to A, $M \cap \kappa \in \kappa$ is inaccessible, $x \in M$ and $[M \cap \lambda]^{<M\cap\kappa} \subseteq M$.*

5.3 Small Models and Forcing

In this subsection we explore the connection between small elementary submodels of $H(\Theta)$ and forcing and reprove well-known folklore results for completeness.

Definition 5.3.1. Let M be a set, $\mathbb{P}$ a forcing order and G a filter on $\mathbb{P}$. We let

$$M[G] := \{\sigma^G \mid \sigma \in M \text{ is a } \mathbb{P}\text{-name}\}$$

We note that if $\mathbb{P} \in M$ and M is an elementary submodel of a sufficiently large $H(\Theta)$, $\Gamma_{\mathbb{P}} \in M$ and so $\Gamma_{\mathbb{P}}^G = G \in M[G]$. By similar reasoning, if $M \prec H(\Theta)$, $M \subseteq M[G]$.

Now we collect various useful results. We start with a statement due to Shelah (which he deemed "confusing to many"; see [She17, chapter III, Theorem 2.11]):

Lemma 5.3.2. *Let $\mathbb{P} \in M \prec H(\Theta)$ and let G be $\mathbb{P}$-generic. Then $M[G] \prec H(\Theta)^{V[G]}$.*

Proof. Since $\mathbb{P} \in H(\Theta)$, $H(\Theta)^{V[G]} = H(\Theta)[G]$. Let $\phi(x, y_1, \ldots, y_n)$ be any formula such that $H(\Theta)[G] \models \exists x \phi(x, y_1, \ldots, y_n)$ for $y_1, \ldots, y_n \in M[G]$. We aim to show there exists $x \in M[G]$ such that $H(\Theta)[G] \models \phi(x, y_1, \ldots, y_n)$ (which suffices by Tarski's Criterion). So let $\tau_1, \ldots, \tau_n \in M$ be $\mathbb{P}$-names with $\tau_i^G = y_i$. By the maximum principle there is a $\mathbb{P}$-name σ such that

$$\emptyset \Vdash \sigma \in H(\Theta)^{V[\Gamma]} \wedge H(\Theta)^{V[\Gamma]} \models \exists x \phi(x, \tau_1, \ldots, \tau_n) \longrightarrow \phi(\sigma, \tau_1, \ldots, \tau_n)$$

We can assume without loss of generality that $\sigma \in H(\Theta)$. Since $\mathbb{P} \in H(\Theta)$,

$$H(\Theta) \models \exists \sigma \left(\emptyset \Vdash H(\Theta)^{V[\Gamma]} \models \exists x \phi(x, \tau_1, \ldots, \tau_n) \longrightarrow \phi(\sigma, \tau_1, \ldots, \tau_n) \right)$$

As $M \prec H(\Theta)$ we can find such a $\sigma \in M$. However, since $H(\Theta)[G] \models \exists x \phi(x, y_1, \ldots, y_n)$, $H(\Theta)[G] \models \phi(\sigma^G, y_1, \ldots, y_n)$, which was what we wanted to show. $\square$

After proving the previous statement, he also notes:

Lemma 5.3.3. *Assume $\mathbb{P} \in M \prec H(\Theta)$ and G is a $\mathbb{P}$-generic filter. Then we have $(M[G], M, \in) \prec (H(\Theta), H^V(\Theta), \in)$ if and only if $M[G] \cap H^V(\Theta) = M$.*

Generalizing the definition (and results) of properness to cardinals larger than ω_1 has been an important set-theoretical subject for some time. However, many results do generalize straightforwardly:

Lemma 5.3.4. *Let $M \prec H(\Theta)$ of size $< \kappa$ with Mostowski-Collapse $\pi \colon M \to N$. Let $\mathbb{P} \in M$ be a forcing order and G a $\mathbb{P}$-generic filter. The following are equivalent:*

1. *$M[G] \cap V = M$*
2. *Whenever $D \in M$ is open dense in $\mathbb{P}$, $M \cap G \cap D \neq \emptyset$.*
3. *Whenever $D \in N$ is open dense in $\pi(\mathbb{P})$, $\pi[M \cap G] \cap D \neq \emptyset$.*

Proof. We show that (1)$\leftrightarrow$(2) and (2)$\leftrightarrow$(3).

Assume $M[G] \cap V = M$ holds and $D \in M$ is open dense. By elementarity, there is in M a maximal antichain $A \subseteq D$. Furthermore, M can construct the name $\tau := \{(p, \check{p}) \mid p \in A\}$ and of course, $\tau^G \in M[G] \cap V$, so $\tau^G \in M$. Since τ^G is the point at which G meets A, $M \cap G \cap A \neq \emptyset$ and thus $M \cap G \cap D \neq \emptyset$.

Now assume $D \cap M \cap G \neq \emptyset$ for every $D \in M$ that is open dense in $\mathbb{P}$. Let τ be a $\mathbb{P}$-name for an element of V. Thus the set (which lies in M) of conditions p such that for some y_p, $p \Vdash \tau = \check{y}_p$, is open dense and is met by $G \cap M$. By elementarity, y_p (which equals τ^G) is in M as well.

The equivalence between (2) and (3) follows because being open dense is absolute between transitive models of enough set theory, so $D \in M$ is open dense in $\mathbb{P}$ if and only if $\pi(D)$ is open dense in $\pi(\mathbb{P})$ (from the perspective of V). $\square$

Lemma 5.3.5. *Let $M \prec H(\Theta)$ of size $<\kappa$ with Mostowski-Collapse $\pi\colon M \to N$. Assume $\nu := M \cap \kappa \in \kappa$, $\mathbb{P} \in M$ is a poset such that $\pi(\mathbb{P})$ has the ν-c.c. and $\mathbb{P} \subseteq X$ such that $[M \cap X]^{<\nu} \subseteq M$. Then the following holds:*

1. *$\pi[G \cap M]$ is $\pi(\mathbb{P})$-generic over V*
2. *$M[G] \cap V = M$,*
3. *$[\pi[X \cap M]]^{<\nu} \cap V[\pi[G \cap M]] \subseteq N[\pi[G \cap M]]$.*

Proof. If $A \subseteq \pi(\mathbb{P})$ is a maximal antichain, $\pi^{-1}[A] \subseteq \mathbb{P} \cap M$ and has size $<\nu$ by the ν-c.c. Therefore $\pi^{-1}[A] \in M$ and $\pi^{-1}[A] \subseteq M$. As $\pi(\pi^{-1}[A]) = A$, $\pi^{-1}[A]$ is a maximal antichain in $\mathbb{P}$ and so there is $p \in \pi^{-1}[A] \cap G$ (which implies $p \in G \cap M$). Thus $\pi(p) \in A \cap \pi[G \cap M]$.

By (1), $\pi[G \cap M]$ is $\pi(\mathbb{P})$-generic over N, by Lemma 5.3.4 this implies $M[G] \cap V = M$.

Now assume τ is a $\pi(\mathbb{P})$-name for an α-sequence of elements of $\pi[X \cap M]$ with $\alpha < \nu$. For $\beta < \alpha$, let A_β be a maximal antichain of conditions $p \in \pi(\mathbb{P})$ such that $p \Vdash \tau(\check{\beta}) = \check{x}_p$ for some $x_p \in \pi[X \cap M]$ and let $f(\beta) := \{\pi^{-1}(p, x_p) \mid p \in A_\beta\}$. By the ν-c.c. of $\pi(\mathbb{P})$, f can be coded as an element of $[M \cap X]^{<\nu}$, so f is in M and $\pi(f) \in N$. However, from $\pi(f)$ and $\pi[G \cap M]$ we can recreate $\tau^{\pi[G \cap M]}$, so $\tau^{\pi[G \cap M]} \in N[\pi[G \cap M]]$. □

We will use the previous result in two ways: If $[M \cap \lambda]^{<\nu} \subseteq M$ and $\mathbb{P}$ is a poset of size $\leq \lambda$, $[\pi(\lambda)]^{<\nu} \cap V[\pi[G \cap M]] \subseteq N[\pi[G \cap M]]$. The other is when M is closed under $<\nu$-sequences of elements of M. In this case we let $X := H(\Theta)$ where Θ is very large and obtain $[N]^{<\nu} \cap V[\pi[G \cap M]] \subseteq N[\pi[G \cap M]]$ which implies that $N[\pi[G \cap M]]$ is closed under $<\nu$-sequences in $V[\pi[G \cap M]]$ by Lemma 5.1.4.

The last result we need can be seen as dual to the lifting of elementary embeddings.

Lemma 5.3.6. *Let $M \prec H(\Theta)$ with Mostowski-Collapse (N, π). Assume $\mathbb{P} \in M$ is a poset, G a $\mathbb{P}$-generic filter and $M[G] \cap V = M$. Then the Mostowski-Collapse of $M[G]$ is given by $(N[\pi[G \cap M]], \pi_{M[G]})$, where $\pi_{M[G]}(\sigma^G) = (\pi(\sigma))^{\pi[G \cap M]}$. Furthermore, $\pi_{M[G]} \restriction M = \pi$.*

Proof. Define $\pi_{M[G]}\colon M[G] \to N[\pi[G \cap M]]$ by $\pi_{M[G]}(\sigma^G) = (\pi(\sigma))^{\pi[G \cap M]}$. This is well-defined and an isomorphism because $M \cap G$ is $\mathbb{P}$-generic over M. We are done after showing that $N[\pi[G \cap M]]$ is transitive because the Mostowski-Collapse is unique.

To this end, let $\sigma_0^{\pi[G \cap M]} \in \sigma_1^{\pi[G \cap M]}$ for $\sigma_1 \in N[\pi[G \cap M]]$. By the definition there exists $p \in \pi[G \cap M]$ and τ with $(p, \tau) \in \sigma_1$ and $\tau^{\pi[G \cap M]} = \sigma_0^{\pi[G \cap M]}$. Because N is transitive, $\tau \in N$, so $\sigma_0^{\pi[G \cap M]} \in N[\pi[G \cap M]]$.

We also show $\pi_{M[G]} \restriction M = \pi$ by induction on rank. Assume $\pi_{M[G]}(x) = \pi(x)$ for every $x \in y \cap M$ with $y \in M$ (note that $y \in M$ implies $y \cap M[G] = y \cap M$ since $y \subseteq V$). Then

$$\pi_{M[G]}(y) = \{\pi_{M[G]}(x) \mid x \in y \cap M[G]\} = \{\pi(x) \mid x \in y \cap M\} = \pi(y)$$

□

5.4 Laver Diamonds for Small Large Cardinals

Another important technique for working with large cardinals is the usage of a Laver function which is a function which is able to anticipate every possible set:

Definition 5.4.1. Let κ be a cardinal and $f\colon \kappa \to V_\kappa$. f is a *Laver function for* κ if for every x and every $\lambda \geq |\operatorname{tcl}(x)|$ there is an elementary embedding $j\colon V \to M$ with critical point κ such that $j(\kappa) > \lambda$, ${}^\lambda M \subseteq M$ and $j(f)(\kappa) = x$.

As has been observed by Laver (see [Lav78]), supercompact cardinals always carry Laver functions:

Theorem 5.4.2. *If κ is supercompact, there exists a Laver function for κ.*

Sadly, the existence of a Laver function for κ requires that κ is a supercompact cardinal and thus carries high consistency strength. However, as has been observed e.g. by Hamkins in [Ham02], we can consistently obtain similar functions at smaller large cardinals using Woodin's Fast Function Forcing.

Definition 5.4.3. Let $\kappa \leq \lambda$ be cardinals such that κ is λ-ineffable. A function $l\colon \kappa \to V_\kappa$ is a *λ-ineffable Laver diamond at κ* if for every large enough Θ, every $A, x \in H(\Theta)$ and every (κ, λ)-list f, there is a λ-ineffability witness M for κ with respect to f such that $A, x \in M$ and (letting π be the Mostowski-Collapse) $l(\pi(\kappa)) = \pi(A)$.

We also have Laver diamonds for Mahlo cardinals:

Definition 5.4.4. Let κ be a cardinal. A function $l\colon \kappa \to V_\kappa$ is a *Mahlo Laver diamond at* κ if for every Θ and every $A, x \in H(\Theta)$ there is $M \prec H(\Theta)$ such that $|M| = M \cap \kappa \in \kappa$ is inaccessible, ${}^{<M\cap\kappa}M \subseteq M$, $x \in M$ and (letting π be the Mostowski-Collapse of M) $l(\pi(\kappa)) = \pi(A)$.

Interestingly, these functions mirror the "standard" Laver diamond in that they are able to guess any set (in contrast, the Laver diamond for weakly compact cardinals as defined by Hamkins only guesses sets of hereditary size $\leq \kappa$).

It has been shown by Hamkins in [Ham02] that Woodin's Fast Function Forcing can add Laver diamonds to many embedding-based large cardinals. The same is true in this context.

Definition 5.4.5. The *Fast Function Forcing at κ*, denoted $\mathbb{F}_\kappa$, consists of partial functions $p\colon \kappa \to \kappa$ such that for all regular $\delta \leq \kappa$, $|p \restriction \delta| < \delta$ and if $\delta \in \operatorname{dom}(p)$, δ is inaccessible and $p[\delta] \subseteq \delta$.

Note that fast function forcing is not κ-c.c. (if κ has κ many inaccessibles below it): Let $(\delta_i)_{i<\kappa}$ enumerate the inaccessibles below κ. For $i < \kappa$, let p_i be a function with domain $\{\delta_0\}$ such that $p_i(\delta_0) = \delta_i$. Then $\{p_i \mid i < \kappa\}$ is a κ-sized antichain.

The main point of fast function forcing is that we can choose how to decompose the forcing: For $\delta < \kappa$ inaccessible, let $\mathbb{F}_{\delta,\kappa}$ consist of all $p \in \mathbb{F}_\kappa$ with $\operatorname{dom}(p) \subseteq [\delta, \kappa)$. If $p := \{(\gamma, \alpha)\}$, $\mathbb{F}_\kappa \restriction p$ is isomorphic to $\mathbb{F}_\gamma \times \mathbb{F}_{\alpha^*,\kappa}$, where α^* is the smallest inaccessible $> \alpha$. Furthermore, $\mathbb{F}_\gamma$ is γ^+-c.c. (because of its size) and $\mathbb{F}_{\alpha^*,\kappa}$ is $<\alpha^*$-closed.

Lemma 5.4.6. *Let G be $\mathbb{F}_\kappa$-generic. If $p = \{(\delta, \alpha)\} \in G$ for some δ, α, then $G \cap \mathbb{F}_\delta$ is $\mathbb{F}_\delta$-generic and moreover, every $<\alpha^*$-sequence of ordinals in $V[G]$ is in $V[G \cap \mathbb{F}_\delta]$.*

Proof. By the preceding paragraph, $\mathbb{F}_\kappa \upharpoonright p$ is isomorphic to $\mathbb{F}_\delta \times \mathbb{F}_{\alpha^*,\kappa}$ via the map $q \mapsto (q \upharpoonright \delta, q \upharpoonright [\alpha^*, \kappa))$. Hence $\{q \upharpoonright \delta \mid q \in G\}$ is $\mathbb{F}_\delta$-generic, but the former set is easily seen to be equal to $G \cap \mathbb{F}_\delta$. Furthermore, $\mathbb{F}_{\alpha^*,\kappa}$ is $<\alpha^*$-distributive in $V[G \cap \mathbb{F}_\delta]$ by the Easton lemma, so it does not add any new $<\alpha^*$-sequences of ordinals. $\square$

This implies (the proof is in [Ham98]):

Lemma 5.4.7. *Let G be $\mathbb{F}_\kappa$-generic. For every cardinal δ, $\mathrm{cf}(\delta)^V = \mathrm{cf}(\delta)^{V[G]}$ and we have $(2^\delta)^V = (2^\delta)^{V[G]}$ (in terms of size).*

Even though $\mathbb{F}_\kappa$ is not κ-c.c., we can perform similar techniques:

Lemma 5.4.8. *Let $M \prec H(\Theta)$ with $\nu := M \cap \kappa \in \kappa$ and $[\nu]^{<\nu} \subseteq M$. If $p := \{(\nu, \alpha)\}$ for some α and G is $\mathbb{F}_\kappa$-generic with $p \in G$, $M[G] \cap V = M$.*

Proof. By Lemma 5.3.4 it suffices to show that $G \cap M \cap D$ is nonempty for every $D \in M$ that is open dense in $\mathbb{F}_\kappa$, so let D be such a set.

We first show that $D \cap \mathbb{F}_\nu$ is open dense in $\mathbb{F}_\nu$: To this end, let $p' \in \mathbb{F}_\nu$. By the closure of M, $p' \in M \cap \mathbb{F}_\kappa$, so there is $q \in D \cap M$ below p'. Because $\sup(\mathrm{im}(q)), \sup(\mathrm{dom}(q)) \in M$, $q \in \mathbb{F}_\nu$.

Because $\nu \in \mathrm{dom}(p)$, the function $\pi\colon q \mapsto q \upharpoonright \nu$ is a projection from $\mathbb{F}_\kappa \upharpoonright p$ to $\mathbb{F}_\nu$. Because G is in particular $\mathbb{F}_\kappa \upharpoonright p$-generic, $\pi[G]$ (which is a subset of G) is $\mathbb{F}_\nu$-generic. Hence $G \cap D \cap \mathbb{F}_\nu$ is nonempty. Lastly, $D \cap \mathbb{F}_\nu = D \cap M$ by the closure of M. $\square$

We finish our preliminary results by showing that, even though $\mathbb{F}_\kappa$ does not have the κ-c.c., every small set is added by a κ-c.c. forcing:

Lemma 5.4.9. *Let ν be a limit of inaccessible cardinals and G a $\mathbb{F}_\nu$-generic filter. If $f \in V[G]$ is a function with domain $<\nu$ and range contained in V there is $\gamma \in \nu$ such that $f \in V[G \cap \mathbb{F}_\gamma]$.*

Proof. Let $\dot{f}$ be a $\mathbb{F}_\nu$-name for such a function and let $\beta = \mathrm{dom}(\dot{f})$ (we assume WLOG that $\emptyset$ decides this, otherwise we work below some condition). The set of conditions q such that some $\gamma > \beta$ is in $\mathrm{dom}(q)$ and $q(\gamma) > \gamma$ is dense. Below any such condition $\mathbb{F}_\nu$ factors as $\mathbb{F}_\gamma \times \mathbb{F}_{q(\gamma)^*,\nu}$. By the Easton lemma q forces that $\dot{f}$ is in $V[\Gamma \cap \mathbb{F}_\gamma]$. $\square$

Now we show that $\mathbb{F}_\kappa$ adds Laver diamonds to Mahlo and λ-ineffable cardinals.

Lemma 5.4.10. *Let κ be a Mahlo cardinal and G an $\mathbb{F}_\kappa$-generic filter. In $V[G]$ there is a Mahlo Laver diamond l.*

Proof. Fix an enumeration $V_\kappa = \{z_\alpha \mid \alpha < \kappa\}$. Let $\dot{l}$ be an $\mathbb{F}_\kappa$-name for a function from κ into V_κ such that $\dot{l}(\alpha) = z_{p(\alpha)}^{G\cap\mathbb{F}_\alpha}$ if there is $p \in G$ with $\alpha \in \mathrm{dom}(p)$ and $z_{p(\alpha)}$ is an $\mathbb{F}_\alpha$-name and $\dot{l}(\alpha) = \emptyset$ otherwise. We want to show that $\dot{l}$ is forced to be a Mahlo Laver diamond at κ.

To this end, let $\Theta > \kappa$ be arbitrary. Let $p \in \mathbb{F}_\kappa$ force $\dot{x}, \dot{A} \in H(\Theta)$ and assume that the names $\dot{x}, \dot{A}$ are in $H(\Theta)$. By Mahloness, fix $M \prec H(\Theta)$ with $\nu := |M| = M \cap \kappa$ an inaccessible cardinal below κ, ${}^{<\nu}M \subseteq M$ and $p, \dot{x}, \dot{A} \in M$. Since $\mathrm{otp}(p) \in M$ and $M \cap \kappa$ is an ordinal, $\mathrm{otp}(p) < \nu$ and $p \subseteq M$. Let $\pi(\dot{A}) = z_\alpha$ for some α and define $q := p \cup \{(\nu, \alpha)\}$. By elementarity $\pi(\dot{A})$ is an $\mathbb{F}_\nu$-name. We want to show that $M[G]$ is as required. Because $M[G] \cap V = M$, $\pi_{M[G]} \upharpoonright M = \pi$, so we denote $\pi := \pi_{M[G]}$.

Clearly $\dot{x}^G \in M[G]$ and $l(\pi(\kappa)) = z_\alpha^{G\cap\mathbb{F}_\alpha} = \pi(\dot{A})^{G\cap\mathbb{F}_\alpha} = \pi(\dot{A}^G)$ (see Lemma 5.3.6). Additionally $|M[G]| = |M| = M \cap \kappa = M[G] \cap \kappa$ and $M[G] \prec H(\Theta)^{V[G]}$.

The last thing left to show is ${}^{<\nu}M[G] \subseteq M[G]$ (in $V[G]$). By Lemma 5.1.4, it suffices to show that ${}^{<\nu}(M[G] \cap \mathrm{On}) = {}^{<\nu}(M \cap \mathrm{On}) \subseteq M[G]$ in $V[G \cap \mathbb{F}_\nu]$. Whenever $f \in {}^{<\nu}(M \cap \mathrm{On})$ is in $V[G \cap \mathbb{F}_\nu]$, there is $\gamma < \nu$ such that $f \in V[G \cap \mathbb{F}_\gamma]$ using Lemma 5.4.9 (by elementarity the class of inaccessibles is unbounded in $M \cap \kappa$ according to M and this is absolute since we do not require stationarity and M is sufficiently closed). Then we can use the γ^+-c.c. of $\mathbb{F}_\gamma$ to find a sequence $(A_\mu)_{\mu<\beta}$ in M where A_μ is a maximal antichain of conditions deciding $\dot{f}(\check{\mu})$. It follows that $\dot{f}^G \in M[G]$. □

A similar proof shows the consistency of λ-ineffable Laver diamonds.

Lemma 5.4.11. *Let $\kappa \leq \lambda = \lambda^{<\kappa}$ be cardinals such that κ is λ-ineffable. Let G be $\mathbb{F}_\kappa$-generic. In $V[G]$, there is a λ-ineffable Laver diamond at κ.*

Proof. First of all, fix an enumeration $V_\kappa = \{z_\alpha \mid \alpha < \kappa\}$ such that every element occurs κ many times and the Gödel pairing function $\langle\cdot,\cdot\rangle$. Let $\dot{l}$ be an $\mathbb{F}_\kappa$-name for a function from κ into V_κ such that $\dot{l}(\alpha) = z_{p(\alpha)}^{G\cap\mathbb{F}_\alpha}$ if there is $p \in G$ with $\alpha \in \mathrm{dom}(p)$ and the definition makes sense (i.e. $z_{p(\alpha)}$ is an $\mathbb{F}_\alpha$-name) and $\dot{l}(\alpha) = \emptyset$ otherwise.

We want to show that $\dot{l}$ is forced to be a λ-ineffable Laver diamond at κ. Let Θ be a cardinal. Let $\dot{f}, \dot{x}, \dot{A}$ be names for a (κ, λ)-list and elements of $H(\Theta)$ respectively (forced by some $p \in \mathbb{F}_\kappa$). By the κ^+-c.c. we can assume that actually $\dot{A}, \dot{x} \in H(\Theta)$. We will transform $\dot{f}$ into a ground-model (κ, λ)-list e. To this end, let $a \in [\lambda]^{<\kappa}$.

1. Assume the following holds: There is $M \prec H(\Theta)$ such that $M \cap \lambda = a$, $\nu := a \cap \kappa$ is an inaccessible cardinal, $\dot{A} \in M$, $\dot{x} \in M$ and there is a $\mathbb{F}_\nu$-name $\dot{x}_a$ as well as a condition $q_a \leq p$ such that $q_a \Vdash \dot{f}(\check{a}) = \dot{x}_a$ and $\pi(\dot{A}) = z_{q_a(\nu)}$. In this case, let

   $$e(a) := \{\langle\alpha, \beta\rangle \in a \mid \alpha \Vdash \check{\beta} \in \dot{x}_a\}$$

   $e(a) \subseteq a$ because M is sufficiently elementary and thus closed under $\langle\cdot,\cdot\rangle$.

2. Otherwise, $e(a) := \emptyset$.

Let $M \prec H(\Theta)$ be a λ-ineffability witness for κ with respect to e such that $p, \dot{x}, \dot{A} \in M$ and $M \cap \kappa$ is an inaccessible cardinal. As before, $p \subseteq M$. We will show that we are in the first case for $a := M \cap \lambda$. First of all, let $\nu := M \cap \kappa$ and $\alpha > \mathrm{otp}(M \cap \lambda)$ be an ordinal such that $\pi(\dot{A}) = z_\alpha$. Consider the condition $q := p \cup \{(\nu, \alpha)\}$. q forces every $< \alpha^*$-sequence of ordinals to be in the extension by $\mathbb{F}_\nu$. Hence there exist q_a and $\dot{x}_a$ as required.

We want to show that q_a forces $M[\Gamma]$ to be a λ-ineffability witness for κ with respect to $\dot{f}$ such that $\dot{l}(M[\Gamma] \cap \kappa) = \pi_{M[\Gamma]}(\dot{A}^G)$. To this end, fix an $\mathbb{F}_\kappa$-generic filter G containing q_a. In particular, $\{(\nu, \alpha)\} \in G$, so $M[G] \cap V = M$ and the Mostowski-Collapse $\pi \colon M \to N$ extends to $\pi \colon M[G] \to N[\pi[G \cap \mathbb{F}_\nu]]$ and $M[G] \prec H(\Theta)^{V[G]}$.

First of all, $z_{q_a(M[G] \cap \kappa)} = z_{q_a(\nu)} = \pi(\dot{A})$ and $\pi(\dot{A})^{G \cap \mathbb{F}_\nu} = \pi(\dot{A}^G)$.

Because M is a λ-ineffability witness for κ with respect to e, there is $b_e \in M$ such that $b_e \cap M = e(M \cap \lambda)$. Let b_f be defined by

$$b_f := \{\beta \in \lambda \mid \exists \alpha \in G \ \langle \alpha, \beta \rangle \in b_e\}$$

clearly $b_f \in M[G]$, so the only thing left to verify is $\dot{f}^G(M[G] \cap \lambda) = \dot{f}^G(M \cap \lambda) = b_f \cap M$.

Let $\beta \in \dot{f}^G(M \cap \lambda)$. By the choice of q_a, $\beta \in \dot{x}_a^{G \cap \mathbb{F}_\nu}$ so there is $\alpha \in G \cap \mathbb{F}_\nu \subseteq M$ such that $\alpha \Vdash \check{\beta} \in \dot{x}_a$. Hence $\beta \in b_f$.

Let $\beta \in b_f \cap M$. By elementarity there is $\alpha \in M \cap G$ such that $\langle \alpha, \beta \rangle \in b_e$. Because M is closed under $\langle \cdot, \cdot \rangle$, $\langle \alpha, \beta \rangle \in M$ and thus in $e(M \cap \lambda)$. By the definition of e, $\alpha \Vdash \check{\beta} \in \dot{x}_a$, so $\beta \in \dot{f}^G(M \cap \lambda)$ by the choice of q_a.

Hence we have shown that underneath every condition there is another condition forcing the existence of a λ-ineffability witness for κ with respect to $\dot{f}$ satisfying the additional constraints. $\square$

Remark 5.4.12. With the λ-ineffable Laver diamond, we can reduce the large cardinal assumptions for $\bigwedge_{n \in \omega} \mathrm{TP}(\omega_{n+2})$ to a sequence $(\kappa_n)_{n \in \omega}$ such that every κ_n is $(\sup_n \kappa_n)^+$-ineffable.

Remark 5.4.13. If l is a λ-ineffable Laver diamond at κ, e a (κ, λ)-list and A arbitrary, we can find a λ-ineffability witness M with $l(M \cap \kappa) = \pi(A)$ satisfying the extra conditions in Lemma 5.2.4 or Lemma 5.2.7, because we obtained those through defining a different (κ, λ)-list.

If l is a supercompact Laver diamond, l is also a λ-ineffable Laver diamond for any $\lambda \geq \kappa$:

Lemma 5.4.14. *Let κ be a supercompact cardinal and $l \colon \kappa \to V_\kappa$ a Laver function. Then the following holds: For any $\lambda \geq \kappa$, any (κ, λ)-list f, any $A \in H(\lambda^+)$, any large enough Θ and any $x \in H(\Theta)$ there exists a λ-ineffability witness N for κ with respect to f such that $x, A \in N$ and, letting π be the Mostowski-Collapse of N, $l(\pi(\kappa)) = \pi(A)$.*

Proof. Let f be a (κ, λ)-list. Let $A \in H(\lambda^+)$. Let $\delta := |H(\lambda^+)|$ and $j \colon V \to W$ an elementary embedding such that $j(\kappa) > \delta$, ${}^\delta W \subseteq W$ and $j(l)(\kappa) = A$. By the closure properties

$j[\lambda] \in W$, so $b := j^{-1}[j(f)(j[\lambda])]$ is defined. Inside V, let Θ be large and $x \in H(\Theta)$. Then the set C' of all $M \prec H(\Theta)$ of size $< \kappa$ containing b,f,κ,λ,A is a club. Hence the set

$$C := \{M \cap H(\lambda^+) \mid M \in C'\}$$

contains a club, so $j[H(\lambda^+)] \in j(C)$. Thus there exists $M \prec H(j(\Theta))^W$ containing $j(b)$, $j(f)$, $j(\kappa)$, $j(\lambda)$, $j(A)$ with $M \cap j(H(\lambda^+)) = j[H(\lambda^+)]$. In particular, $M \cap j(\lambda) = j[\lambda]$, so

$$j(b) \cap (M \cap j(\lambda)) = j(b) \cap j[\lambda] = j(f)(j[\lambda])$$

and M is a $j(\lambda)$-ineffability witness for $j(\kappa)$ with respect to $j(f)$. Let (N, π) be its Mostowski-Collapse.

Claim. *If $x \in H(\lambda^+)$, $\pi(j(x)) = x$.*

Proof. Note that the statement makes sense because if $x \in H(\lambda^+)$, $j(x) \in j[H(\lambda^+)] \subseteq M$.

We prove it by $\in$-induction. Assume $y \in H(\lambda^+)$ and $\pi(j(x)) = x$ for every $x \in y$. By the definition

$$\pi(j(y)) = \{\pi(a) \mid a \in j(y) \cap M\}$$

Assume $a \in j(y) \cap M$. Because $j(y) \in j(H(\lambda^+))$, $a \in M \cap j(H(\lambda^+)) = j[H(\lambda^+)]$, so there is $x \in H(\lambda^+)$ with $j(x) = a$. By assumption $x \in y$ so $\pi(a) = \pi(j(x)) = x \in y$.

Assume $x \in y$. Then $j(x) \in j(y)$ and $j(x) \in j[H(\lambda^+)]$, so $j(x) \in M$. It follows that $\pi(j(x)) = x \in \pi(j(y))$. $\square$

Thus $j(l)(\pi(j(\kappa))) = j(l)(\kappa) = A = \pi(j(A))$. In summary, in W, there exists a $j(\lambda)$-ineffability witness for $j(\kappa)$ with respect to $j(f)$ such that $j(l)(\pi(j(\kappa))) = \pi(j(A))$.

By elementarity there is in V a λ-ineffability witness for κ with respect to f such that $l(\pi(\kappa)) = \pi(A)$. $\square$

CHAPTER 6

A General Mitchell Forcing

In the following two chapters, we will construct many variants of Mitchell Forcing (as the thesis name suggests). These variants all share common techniques (a projection analysis and the fact that the quotients by initial segments resemble the original forcing). In this chapter, we will introduce a general framework which can be used to construct variants of Mitchell Forcing to simplify the later proofs. This is unpublished work due to the author.

6.1 Defining $\mathbb{M}$

We define our general Mitchell Forcing. This definition resembles the idea of a *mixed support iteration* introduced by Krueger in [Kru08a]. However, we change the forcing in the following three ways:

1. By working with forcings which more closely resemble Mitchell's original poset, our forcing comes with a projection analysis which is crucial when trying to obtain results for successive cardinals.
2. We allow a "decoupling" of the type of Cohen forcing and the closure of the collapses, allowing us to obtain specific results regarding the exact degree of slenderness necessary to guarantee the existence of ineffable branches.
3. We allow Cohen forcings and closed forcings at arbitrary points, allowing us to construct guessing variants of Mitchell Forcing.

Definition 6.1.1. Let $\tau < \mu < \kappa$ be regular cardinals. Let $A \subseteq \kappa$ and $F\colon \kappa \to V_\kappa$. The forcing $\mathbb{M}(\tau,\mu,\kappa,A,F,\beta)$ is defined by induction on $\beta \leq \kappa$.

We let $\mathbb{M}(\tau,\mu,\kappa,A,F,0) = \{\emptyset\}$.

Assume $\mathbb{M}(\tau,\mu,\kappa,A,F,\gamma)$ has been defined for all $\gamma < \beta$. Let $\mathbb{M}(\tau,\mu,\kappa,A,F,\beta)$ consist of conditions (p,q) where

1. $p \in \mathrm{Add}(\tau, A \cap \beta)$,
2. q is a partial function on β of size $<\mu$ and for every $\gamma \in \mathrm{dom}(q)$, $F(\gamma)$ is an $\mathbb{M}(\tau,\mu,\kappa,A,F,\gamma)$-name for a partial order such that the term ordering on the poset $\mathbb{M}(\tau,\mu,\kappa,A,F,\gamma) * F(\gamma)$ is μ-strategically closed and $q(\gamma)$ is an $\mathbb{M}(\tau,\mu,\kappa,A,F,\gamma)$-name for an element of $F(\gamma)$.

We let $(p', q') R_{\mathbb{M}(\tau,\mu,\kappa,A,F,\beta)} (p, q)$ if

1. $p' \leq p$ in $\mathrm{Add}(\tau, A \cap \beta)$,
2. $\mathrm{dom}(q') \supseteq \mathrm{dom}(q)$ and for each $\gamma \in \mathrm{dom}(q)$,

$$(p' \restriction \gamma, q' \restriction \gamma) \Vdash q'(\gamma) \leq_{F(\gamma)} q(\gamma)$$

We set $\mathbb{M}(\tau, \mu, \kappa, A, F) := \mathbb{M}(\tau, \mu, \kappa, A, F, \kappa)$.

For simplicity, we define the following:

Definition 6.1.2. We let $\mathrm{supp}(F)$ consist of all those γ such that $F(\gamma)$ is an $\mathbb{M}(\tau, \mu, \kappa, A, F, \gamma)$-name for a partial order such that the term ordering on the poset $\mathbb{M}(\tau, \mu, \kappa, A, F, \gamma) * F(\gamma)$ is μ-strategically closed.

Remark 6.1.3. We note the following:

1. Since the definition of $\mathbb{M}(\tau, \mu, \kappa, A, F, \gamma)$ only depends on $A \cap \gamma$ and $F \restriction \gamma$, we can define A and F inductively such that the value of $\gamma \in A$ and $F(\gamma)$ depends on $\mathbb{M}(\tau, \mu, \kappa, A, F, \gamma)$.
2. By our definition it is possible that q will never be defined on some ordinals γ (if $F(\gamma)$ is not an $\mathbb{M}(\tau, \mu, \kappa, A, F, \gamma)$-name for a partial order with a μ-strategically closed term ordering). However, this is of course not an issue.
3. The last point in the definition uses that if $(p, q) \in \mathbb{M}(\tau, \mu, \kappa, A, F, \beta)$, its restriction $(p \restriction \gamma, q \restriction \gamma) \in \mathbb{M}(\tau, \mu, \kappa, A, F, \gamma)$ which is clearly true.
4. We can view $(\mathbb{M}(\tau, \mu, \kappa, A, F, \beta), R_{\mathbb{M}(\tau,\mu,\kappa,A,F,\beta)})$ as an ordering on a product of the form $\mathrm{Add}(\tau, A \cap \beta) \times \mathbb{T}(\tau, \mu, \kappa, A, F, \beta)$ which is based (see Definition 4.1.1). Moreover, the induced base ordering is clearly equal to the regular ordering on $\mathrm{Add}(\tau, A \cap \beta)$.
5. Given $(p, q) \in \mathbb{M}(\tau, \mu, \kappa, A, F, \gamma)$, let $(p, q) \restriction \gamma := (p \restriction \gamma, q \restriction \gamma)$.
6. For readability, we will mostly denote $R_{\mathbb{M}(\tau,\mu,\kappa,A,F,\beta)}$ by $\leq$ if the order is clear.

6.2 The Properties of $\mathbb{M}$

For this section, fix regular cardinals $\tau < \mu < \kappa$ as well as $A \subseteq \kappa$ and $F\colon \kappa \to V_\kappa$. Assume further that $\tau^{<\tau} = \tau$ and κ is inaccessible.

Lemma 6.2.1. *Let $\beta \leq \kappa$.*

1. *If β is inaccessible and $F[\beta] \subseteq V_\beta$, $\mathbb{M}(\tau, \mu, \kappa, A, F, \beta)$ is β-Knaster.*
2. *If $F(\gamma)$ is forced to be $<\tau$-closed for every $\gamma \in \beta$, $\mathbb{M}(\tau, \mu, \kappa, A, F, \beta)$ is $<\tau$-closed.*

3. *The base ordering on* $\mathbb{M}(\tau, \mu, \kappa, A, F, \beta)$ *is isomorphic to* $\mathrm{Add}(\tau, A \cap \beta)$*.*

4. *The term ordering on* $\mathbb{M}(\tau, \mu, \kappa, A, F, \beta)$ *is* μ*-strategically closed.*

5. *The ordering has the projection and the refinement property.*

6. *If there is* $\xi \in A$ *such that* $\mathrm{im}(F \upharpoonright [0, \xi]) = \{\check{1}\}$*, the ordering has the mixing property.*

Proof. We only show (1) and (2), the other statements will be shown later in a harder case.

For (1), let $\{(p_\alpha, q_\alpha) \mid \alpha < \beta\}$ be a sequence of conditions. By the Δ-System-Lemma we can assume that there are r_p, r_q and $X \subseteq \beta$ of size β such that $\mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta) = r_p$ and $\mathrm{dom}(q_\alpha) \cap \mathrm{dom}(q_\beta) = r_q$ for all $\alpha, \beta \in X$. Lastly, there are $< \beta$ choices for $p_\alpha \upharpoonright r_p$ and $q_\alpha \upharpoonright r_q$, as $F[\beta] \subseteq V_\beta$ and β is inaccessible. So we can find $Y \subseteq X$ of size β such that all (p_α, q_α) $(\alpha \in Y)$ are compatible.

For (2), let $(p_\alpha, q_\alpha)_{\alpha < \delta}$ $(\delta < \tau)$ be a descending sequence of conditions. We define $p := \bigcup_{\alpha < \delta} p_\alpha$, $x := \bigcup_{\alpha < \delta} \mathrm{dom}(q_\alpha)$ and, by induction, find q with domain x such that $(p, q) \upharpoonright \gamma \leq (p_\alpha, q_\alpha) \upharpoonright \gamma$ for all $\alpha < \delta$.

Assume $q \upharpoonright \gamma$ has been defined and $\gamma \in x$. By the inductive hypothesis, $(p, q) \upharpoonright \gamma$ forces the sequence $(q_\alpha(\gamma))_{\delta_0 \leq \alpha < \delta}$ (where δ_0 is minimal with $\gamma \in \mathrm{dom}(q_{\delta_0})$) to be descending. By the hypothesis and the maximum principle there is $q(\gamma)$ forced to be a lower bound. Then $(p, q) \upharpoonright \gamma + 1$ is below $(p_\alpha, q_\alpha) \upharpoonright \gamma + 1$ for $\alpha < \delta$. $\square$

In particular, we obtain:

Corollary 6.2.2. *Let* $\beta \leq \kappa$*.*

1. *There is a* μ*-strategically closed forcing* $\mathbb{T}$ *and a projection from* $\mathrm{Add}(\tau, A \cap \beta) \times \mathbb{T}$ *to* $\mathbb{M}(\tau, \mu, \kappa, A, F, \beta)$*.*

2. $\mathbb{M}(\tau, \mu, \kappa, A, F, \beta)$ *has the* $<\delta$*-covering property for any* $\delta \in [0, \mu]$

3. *If* $F[\beta] \subseteq \beta$ *and* β *is inaccessible,* $\mathbb{M}(\tau, \mu, \kappa, A, F, \beta)$ *has the* $<\beta$*-covering property.*

Proof. (1) is just Lemma 4.1.3 together with Lemma 6.2.1, (3) and (4).

(2) follows from (1): Any $<\delta$-sized set of ordinals for $\delta \leq \mu$ has been added by $\mathrm{Add}(\tau, A \cap \beta)$. If $\delta \leq \tau$, the set is even in the ground model. If $\delta > \tau$, it is covered by a set of size $<\delta$ by the τ^+-c.c. of $\mathrm{Add}(\tau, \beta)$.

(3) follows directly from Lemma 6.2.1, (1). $\square$

As is common when working with variants of Mitchell forcing, we will explicitely construct a partial order which functions as the quotient forcing for $\mathbb{M}(\tau, \mu, \kappa, A, F)$ by some initial segment $\mathbb{M}(\tau, \mu, \kappa, A, F, \beta)$ (i.e. an $\mathbb{M}(\tau, \mu, \kappa, A, F, \beta)$-name $\dot{\mathbb{Q}}$ for a poset such that $\mathbb{M}(\tau, \mu, \kappa, A, F, \beta) * \dot{\mathbb{Q}}$ is forcing equivalent to $\mathbb{M}(\tau, \mu, \kappa, A, F)$). Given a $\mathbb{P} * \dot{\mathbb{Q}}$-name τ and a $\mathbb{P}$-generic filter G, let τ^G denote a $\dot{\mathbb{Q}}^G$-name such that for every $\dot{\mathbb{Q}}^G$-generic H, $(\tau^G)^H = \tau^{G*H}$.

Definition 6.2.3. Let $\nu < \kappa$ and let G be $\mathbb{M}(\tau,\mu,\kappa,A,F,\nu)$-generic. In $V[G]$, define the partial order $\mathbb{M}(G,\tau,\mu,\kappa\setminus\nu,A,F,\beta)$ by induction on $\beta \in [\nu,\kappa]$, starting by letting $\mathbb{M}(G,\tau,\mu,\kappa\setminus\nu,A,F,\nu) := \{\emptyset\}$. For every β, this defines an $\mathbb{M}(\tau,\mu,\kappa,A,F,\nu)$-name $\mathbb{M}(\Gamma,\tau,\mu,\kappa\setminus\nu,A,F,\beta)$ in V.

Assume that for all $\gamma \in [\nu,\beta)$, $\mathbb{M}(G,\tau,\mu,\kappa\setminus\nu,A,F,\gamma)$ has been defined and there is a dense embedding from $\mathbb{M}(\tau,\mu,\kappa,A,F,\gamma)$ into $\mathbb{M}(\tau,\mu,\kappa,A,F,\nu) * \mathbb{M}(\Gamma,\tau,\mu,\kappa\setminus\nu,A,F,\gamma)$ (otherwise set $\mathbb{M}(G,\tau,\mu,\kappa\setminus\nu,A,F,\beta) := \{\emptyset\}$). Then let $\mathbb{M}(G,\tau,\mu,\kappa\setminus\nu,A,F,\beta)$ consist of pairs (p,q) such that

1. $p \in \mathrm{Add}(\tau, A\cap[\nu,\beta))$,
2. q is a partial function on $\mathrm{supp}(F)\cap[\nu,\beta)$ of size $<\mu$ and for every $\gamma\in\mathrm{dom}(q)$, $q(\gamma)$ is an $\mathbb{M}(G,\tau,\mu,\kappa\setminus\nu,A,F,\gamma)$-name for an element of $F(\gamma)^G$ (viewing $F(\gamma)$ as an $\mathbb{M}(\tau,\mu,\kappa,A,F,\nu) * \mathbb{M}(\Gamma,\tau,\mu,\kappa\setminus\nu,A,F,\gamma)$-name).

We let $(p',q') \leq (p,q)$ if

1. $p' \leq p$ in $\mathrm{Add}(\tau, A\cap[\nu,\beta))$,
2. $\mathrm{dom}(q') \supseteq \mathrm{dom}(q)$ and for each $\gamma\in\mathrm{dom}(q)$,
$$(p'\restriction[\nu,\gamma), q'\restriction[\nu,\gamma)) \Vdash q'(\gamma) \leq_{F(\gamma)} q(\gamma)$$

We define $\mathbb{M}(G,\tau,\mu,\kappa\setminus\nu,A,F) := \mathbb{M}(G,\tau,\mu,\kappa\setminus\nu,A,F,\kappa)$.

Remark 6.2.4. As with $\mathbb{M}(\tau,\mu,\kappa,A,F,\gamma)$, we view $\mathbb{M}(G,\tau,\mu,\kappa\setminus\nu,A,F,\gamma)$ as an ordering on a product $\mathrm{Add}(\tau,A\cap[\nu,\gamma)) \times \mathbb{T}(G,\tau,\mu,\kappa\setminus\nu,A,F,\gamma)$ which is based.

Now we show that this forcing works as intended (so in particular, the construction proceeds until κ).

Lemma 6.2.5. *Let $\nu<\kappa$. For any $\beta\in[\nu,\kappa]$, there is a dense embedding from $\mathbb{M}(\tau,\mu,\kappa,A,F,\beta)$ into $\mathbb{M}(\tau,\mu,\kappa,A,F,\nu) * \mathbb{M}(\Gamma,\tau,\mu,\kappa\setminus\nu,A,F,\beta)$.*

Proof. We do the proof by induction with the base case being clear.

Assume that for all $\gamma<\beta$, there is such an embedding. So in particular, the name $\mathbb{M}(\Gamma,\tau,\mu,\kappa\setminus\nu,A,F,\beta)$ is nontrivial. Define the following function ι:

$$\iota(p,q) := ((p\restriction\nu, q\restriction\nu), \mathrm{op}(\check{p}\restriction[\check{\nu},\check{\beta}), \overline{q})$$

where op maps two names to the name for their ordered pair and $\overline{q}$ is defined as follows: It is an $\mathbb{M}(\tau,\mu,\kappa,A,F,\nu)$-name for a function with domain $\mathrm{dom}(q)\cap[\nu,\beta)$ and for each $\gamma\in\mathrm{dom}(q)\cap(\beta\setminus\nu)$,

$$\Vdash_{\mathbb{M}(\tau,\mu,\kappa,A,F,\nu)*\mathbb{M}(\Gamma,\tau,\mu,\kappa\setminus\nu,A,F,\gamma)} \overline{q}(\check{\gamma}) = q(\gamma)^{\Gamma}$$

We prove the relevant properties one by one:

Claim. *If $(p,q) \in \mathbb{M}(\tau,\mu,\kappa,A,F,\beta)$, $\iota(p,q) \in \mathbb{M}(\tau,\mu,\kappa,A,F,\nu) * \mathbb{M}(\Gamma,\tau,\mu,\kappa \setminus \nu,A,F,\beta)$.*

Proof. Clearly, $(p \restriction \nu, q \restriction \nu) \in \mathbb{M}(\tau,\mu,\kappa,A,F,\nu)$. Let G be $\mathbb{M}(\tau,\mu,\kappa,A,F,\nu)$-generic. Then $(p \restriction [\check{\nu},\beta))^G = p \restriction [\nu,\beta) \in \mathrm{Add}(\tau, A \cap [\nu,\beta))$. By the definition, $\overline{q}^G$ is a partial function on $[\nu,\beta)$ of size $< \mu$ and for every $\gamma \in \mathrm{dom}(\overline{q}^G)$, $\gamma \in \mathrm{dom}(q)$, so $\gamma \in \mathrm{supp}(F)$ and $\overline{q}^G(\gamma) = q(\gamma)^G$ is an $\mathbb{M}(G,\tau,\mu,\kappa \setminus \nu,A,F,\gamma)$-name for an element of $F(\gamma)^G$. □

Similarly, one obtains that ι respects the ordering. Lastly, we have to show:

Claim. *The image of ι is dense.*

Proof. Let $((p_0,q_0),\sigma) \in \mathbb{M}(\tau,\mu,\kappa,A,F,\nu) * \mathbb{M}(\Gamma,\tau,\mu,\kappa \setminus \nu,A,F,\beta)$. By strengthening (p_0,q_0) if necessary, we can assume that $\sigma = \mathrm{op}(\check{p}_1,\dot{q})$ such that for some $x \in V$ of size $< \mu$, $(p_0,q_0) \Vdash \mathrm{dom}(\dot{q}) \subseteq \check{x}$. By strengthening $\dot{q}$ (and assuming $x \subseteq \mathrm{supp}(F)$) if necessary, we can assume that $(p_0,q_0) \Vdash \mathrm{dom}(\dot{q}) = \check{x}$. Let q_1 be the following function with domain x: Given $\gamma \in x$, there is an $\mathbb{M}(\tau,\mu,\kappa,A,F,\nu)$-name τ_γ for an $\mathbb{M}(\Gamma,\tau,\mu,\kappa \setminus \nu,A,F,\gamma)$-name such that $(p_0,q_0) \Vdash \dot{q}(\check{\gamma}) = \tau_\gamma$. Let $q_1(\gamma)$ be the corresponding $\mathbb{M}(\tau,\mu,\kappa,A,F,\gamma)$-name (using the dense embedding that exists by the inductive hypothesis).

Now it is clear that $\iota(p_0 \cup p_1, q_0 \cup q_1) \leq ((p_0,q_0),\sigma)$. □

This finishes the proof. □

Now we show the same properties for $\mathbb{M}(\Gamma,\tau,\mu,\kappa \setminus \nu,A,F,\beta)$. We essentially obtain the statements of Lemma 6.2.1 by setting $\nu := \emptyset$.

Lemma 6.2.6. *Let $\nu \leq \beta \leq \kappa$ and G be $\mathbb{M}(\tau,\mu,\kappa,A,F,\nu)$-generic.*

1. *The base ordering on $\mathbb{M}(G,\tau,\mu,\kappa \setminus \nu,A,F,\beta)$ is isomorphic to $\mathrm{Add}(\tau, A \cap [\nu,\beta))$.*
2. *The ordering has the projection and the refinement property.*
3. *If there is $\xi \in [\nu,\beta) \cap A$ such that $\mathrm{im}(F \restriction [\nu,\xi]) = \{\check{1}\}$, the ordering has the mixing property.*
4. *Assume the term ordering on $\mathbb{M}(\tau,\mu,\kappa \setminus \nu,A,F,\gamma) * F(\gamma)^G$ is μ-strategically closed for every $\gamma \in \mathrm{supp}(F) \cap [\nu,\beta)$. Then the term ordering on $\mathbb{M}(\Gamma,\tau,\mu,\kappa \setminus \nu,A,F,\beta)$ is μ-strategically closed.*

Proof. We prove the statements one by one.

1. This is clear.
2. The refinement property is clear since stronger conditions force more. For the projection property, let $(p',q') \leq (p,q)$. By induction on γ, we define $q''(\gamma)$ with domain $\mathrm{dom}(q')$ such that $(p,q'') \restriction \gamma \leq (p,q) \restriction \gamma$ and

$$(p',q'') \restriction \gamma \leq (p',q') \restriction \gamma \leq (p',q'') \restriction \gamma$$

Assume q'' has been defined until γ and $\gamma \in \operatorname{dom}(q')$. Let $q''(\gamma)$ be a name such that $(p', q') \restriction \gamma$ forces $q''(\gamma) = q'(\gamma)$ and conditions incompatible with $(p', q') \restriction \gamma$ force $q''(\gamma) = q(\gamma)$. It follows that this $q'' \restriction \gamma + 1$ is as required.

3. Let $(p, q_0), (p, q_1) \leq (p, q)$. Let $\xi \in [\nu, \beta) \cap A$ be as in the assumption and p_0, p_1 extensions of p such that $p_0(\xi)$ and $p_1(\xi)$ are incompatible in $\operatorname{Add}(\tau)$. As before, we define q' by induction on γ such that $(p, q') \restriction \gamma \leq (p, q) \restriction \gamma$ and $(p_i, q') \restriction \gamma \leq (p, q_i) \restriction \gamma$.

   Assume $q' \restriction \gamma$ has been defined. If $F(\gamma) = \check{1}$, $\Vdash q_0(\gamma) = \emptyset = q_1(\gamma)$, so let $q'(\gamma) := q_0(\gamma)$. Otherwise, $\gamma > \xi$, so $p_0 \restriction \gamma$ and $p_1 \restriction \gamma$ are incompatible in $\operatorname{Add}(\tau, A \cap [\nu, \gamma))$ and thus $(p_0, q') \restriction \gamma$ and $(p_1, q') \restriction \gamma$ are incompatible in $\mathbb{M}(G, \tau, \mu, \kappa \setminus \nu, A, F, \gamma)$. Ergo we can find $q'(\gamma)$ that is forced by $(p_0, q') \restriction \gamma$ to be equal to $q_0(\gamma)$ and by conditions incompatible with $(p_0, q') \restriction \gamma$ (in particular, $(p_1, q') \restriction \gamma$) to be equal to $q_1(\gamma)$. This $q' \restriction \gamma + 1$ is as required.

4. It suffices to show that the section ordering induced by p (given by $q(s(R, p))q'$ if $(p, q) \leq (p, q')$) is μ-strategically closed for every p. To this end, let p be given. For every $\gamma \in \operatorname{supp}(F) \cap [\nu, \beta)$, fix a winning strategy σ_γ in the completeness game played on the section ordering induced by $(p \restriction \gamma, \emptyset)$ on $\mathbb{M}(G, \tau, \mu, \kappa \setminus \nu, A, F, \gamma) * F(\gamma)^G$. Assume for simplicity that every σ_γ maps the constant $1_{\mathbb{P}}$-sequence to $1_{\mathbb{P}}$.

   Now our strategy σ is defined as follows (by induction on δ): Given $(p, q_\alpha)_{\alpha < \delta}$ (with $\delta < \mu$ and played according to σ), let $x := \bigcup_{\alpha < \delta} \operatorname{dom}(q_\alpha)$ and let q_δ on x be defined as follows: $q_\delta(\gamma) = \sigma_\gamma(((p \restriction \gamma, \emptyset), q_\alpha(\gamma))_{\alpha < \delta})$ (where we let $q_\alpha(\gamma) = 1$ if $\gamma \notin \operatorname{dom}(q_\alpha)$). By induction, $((p \restriction \gamma, \emptyset), q_\alpha(\gamma))_{\alpha < \delta}$ has been played according to σ_γ, so $q_\delta(\gamma)$ exists. Lastly, we show that q_δ satisfies the rules:

   Claim. *For all $\alpha < \delta$, $(p, q_\delta) \leq (p, q_\alpha)$.*

   Proof. Let $\alpha < \delta$ be arbitrary. Clearly, $\operatorname{dom}(q_\delta) \supseteq \operatorname{dom}(q_\alpha)$. Let $\gamma \in \operatorname{dom}(q_\alpha)$. Then $(p \restriction \gamma, \emptyset) \Vdash q_\delta(\gamma) \leq q_\alpha(\gamma)$ and so $(p \restriction \gamma, q_\delta \restriction \gamma) \Vdash q_\delta(\gamma) \leq q_\alpha(\gamma)$. □

   This finishes the proof.

□

By applying Theorem 4.2.2, we obtain:

Corollary 6.2.7. *Assume that for every $\gamma \in \operatorname{supp}(F) \cap [\nu, \kappa)$, the term ordering on the poset $\mathbb{M}(G, \tau, \mu, \kappa \setminus \nu, A, F, \gamma) * F(\gamma)^G$ is μ-strategically closed and there is $\xi \in [\nu, \kappa) \cap A$ such that $\operatorname{im}(F \restriction [\nu, \xi]) = \{\check{1}\}$. Then $\mathbb{M}(G, \tau, \mu, \kappa \setminus \nu, A, F)$ has the $< \tau^+$-approximation property.*

This also means that $\mathbb{M}(G, \tau, \mu, \kappa \setminus \nu, A, F)$ has the $< \delta$-approximation property for any $\delta \geq \tau^+$: If $f \in V[\mathbb{M}(G, \tau, \mu, \kappa \setminus \nu, A, F)]$ is such that $f \restriction x \in V$ for all $x \in [V]^{<\delta}$, then in particular $f \restriction x \in V$ for all $x \in [V]^{<\tau^+}$, so $f \in V$.

Lastly, we state (the proof is clear) that forcing with $\mathbb{M}$ is similar to an iteration:

Lemma 6.2.8. *Let $\gamma < \kappa$ be an ordinal.*

1. *If $\gamma \in A \cap \operatorname{supp}(F)$, $\mathbb{M}(\tau, \mu, \kappa, A, F, \gamma + 1) \cong \mathbb{M}(\tau, \mu, \kappa, A, F, \gamma) * (\operatorname{Add}(\tau) \times F(\gamma))$.*
2. *If $\gamma \in \operatorname{supp}(F) \setminus A$, $\mathbb{M}(\tau, \mu, \kappa, A, F, \gamma + 1) \cong \mathbb{M}(\tau, \mu, \kappa, A, F, \gamma) * F(\gamma)$.*

Remark 6.2.9. Most of these techniques (apart from the closure of $\operatorname{Add}(\tau)$) also work with inner model versions of $\operatorname{Add}(\tau)$ (as has been done e.g. in [CF98]), as long as these versions are square-τ^+-c.c. and $<\tau$-distributive. However, this will not be of use to us since we do not iterate these variants of Mitchell forcing and only work with products.

CHAPTER 7

The Internal Structure of Elementary Submodels

In [FT05], the authors introduced the following four ways a set N with $|N| = \mu$ (μ regular uncountable) can relate to the collection of its $< \mu$-sized subsets:

Definition 7.0.1. Let μ be a regular uncountable cardinal and N a set with $|N| = \mu$.

1. N is *internally unbounded* if $[N]^{<\mu} \cap N$ is unbounded in $[N]^{<\mu}$.
2. N is *internally stationary* if $[N]^{<\mu} \cap N$ is stationary in $[N]^{<\mu}$.
3. N is *internally club* if $[N]^{<\mu} \cap N$ contains a club in $[N]^{<\mu}$.
4. N is *internally approachable* if there exists an $\in$-increasing and continuous sequence $(N_i)_{i<\mu}$ of elements of $[N]^{<\mu}$ such that $\bigcup_{i<\mu} N_i = N$ and for every $j < \mu$, $(N_i)_{i<j} \in N$.

The authors ask if all of these properties are equivalent for all elementary submodels of sufficiently large $H(\Theta)$. This was answered by Krueger in a series of papers (cf. [Kru08c] and [Kru09]) showing that for each of the inclusions there is a forcing extension where it is strict on a stationary set.

Related to the internal structure of elementary submodels is the idea of a disjoint stationary (club) sequence on μ^+, a sequence $(A_\alpha)_{\alpha\in S}$ (where $S \subseteq \mu^+ \cap \mathrm{cof}(\mu)$ is stationary) such that $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$ and each A_α is stationary (club) in $[\alpha]^{<\mu}$. Another related definition is that of the approachability ideal: Given a sequence $\overline{x} = (x_i)_{i<\mu^+}$ of elements of $[\mu^+]^{<\mu}$, a point γ is *approachable with respect to* $\overline{x}$ if there is $A \subseteq \gamma$ unbounded with minimal ordertype such that $\{A \cap \beta \mid \beta < \gamma\} \subseteq \{x_i \mid i < \gamma\}$. A set A is in the *approachability ideal* if there exists a sequence $\overline{x}$ and a club C such that any ordinal in $A \cap C$ is approachable with respect to $\overline{x}$.

In [Kru09], Krueger asks the following questions:

1. Is it consistent that there is a disjoint stationary sequence on ω_{n+2}, for every $n \in \omega$?
2. Is it consistent that the notions of internal stationarity and internal clubness are distinct for stationarily many $N \in [H(\Theta)]^{\omega_{n+1}}$, for every $n \in \omega$ and $\Theta > \omega_{n+1}$?
3. Is it consistent that the notions of internal clubness and internal approachability are distinct for stationarily many $N \in [H(\Theta)]^{\omega_{n+1}}$, for every $n \in \omega$ and $\Theta > \omega_{n+1}$?

4. What is the consistency strength of the assertion that for every $\Theta > \kappa$, the notions of internal stationarity and internal clubness are distinct for stationarily many $N \in [H(\Theta)]^\kappa$?

5. What is the consistency strength of the assertion that for every $\Theta > \kappa$, the notions of internal clubness and internal approachability are distinct for stationarily many $N \in [H(\Theta)]^\kappa$?

In this chapter we will answer all of the above questions. In particular, we will show that the answers to questions (1) to (3) are affirmative and the answer to questions (4) and (5) is "a Mahlo Cardinal" (we will show that a Mahlo cardinal is sufficient; Since a Mahlo cardinal is already necessary for the distinction in $[H(\kappa^+)]^\kappa$, this answers the question). The latter answers provide a dramatic weakening in consistency strength: In his papers, Krueger obtained the global distinction between internal stationarity and internal clubness as well as internal clubness and internal approachability by using models obtained through supercompact embeddings (models of the form $j[H(\Theta)]$ for κ a $|H(\Theta)|$-supercompact cardinal) and collapsed cardinals given by a Laver function (this was necessary because $|j[H(\Theta)]| > \kappa$ for $\Theta > \kappa$). However, he showed that the existence of a disjoint stationary or club sequence (and thus the distinction in $[H(\kappa^+)]^\kappa$) is consistent from merely a Mahlo cardinal.

In contrast to Krueger, we will use models obtained using Mahlo cardinals. These models have the property that $|M| = M \cap \kappa$. Because of this, by simply collapsing $M \cap \kappa$ in the correct way, the whole model M will serve as a witness to the distinction between internal stationarity and internal clubness (or internal clubness and internal approachability).

We will also consider the properties from the following viewpoint: In [For05], Foreman asked if $M \prec H(\omega_3)$ must be internally stationary provided that $M \cap H(\omega_2)$ is internally stationary. While the answer to this question does not fit into the context of this thesis, it prompted an investigation into related problems. We will show that there can consistently be stationarily many $N \in [H(\mu^{++})]^\mu$ such that $N \cap H(\mu^+)$ is internally approachable and N itself is internally club but not internally approachable. We will also show using related techniques that it is consistent that the approachability property holds at μ^+ and there are stationarily many $N \in [H(\mu^+)]^\mu$ which are internally stationary but not internally club. This shows that two results (one folklore, see [Cox21], the other due to Krueger, see [Kru09]) relating AP and DSS to distinctions between variants of internal approachability depend on the assumption $2^\mu = \mu^+$ and answers a question of Levine from [Lev24].

The results about the distinction between internal stationarity and clubness are due to the author (see [Jak23]). The results about the distinction between internal clubness and approachability are joint work with Maxwell Levine (see [JL25]). The results about models which are internally approachable of different types at different levels are again due to the author (see [Jak24a]).

7.1 Preliminary Results

We present some interesting and helpful well-known results related to the properties above.

Lemma 7.1.1. *Let $M \prec H(\Theta)$ be of size μ and let $\pi\colon M \to N$ be its Mostowski-Collapse. Assume $\mu \subseteq M$.*

1. *M is internally unbounded if and only if N is internally unbounded.*
2. *M is internally stationary if and only if N is internally stationary.*
3. *M is internally club if and only if N is internally club.*
4. *M is internally approachable if and only if N is internally approachable.*

Proof. We just show (2), the other statements follow similarly.

Assume M is internally stationary. Let $C \subseteq [N]^{<\mu}$ be a club. Since π is a bijection, $\{\pi^{-1}[x] \mid x \in C\}$ is club in $[M]^{<\mu}$. By internal stationarity of M, there is $x \in C$ with $\pi^{-1}[x] \in M$. This implies $\pi^{-1}[x] \subseteq M$ and thus $\pi(\pi^{-1}[x]) = \pi[\pi^{-1}[x]] = x \in N \cap C$.

Assume N is internally stationary. Let $C \subseteq [M]^{<\mu}$ be a club. Since π is a bijection, $\{\pi[x] \mid x \in C\}$ is club in $[N]^{<\mu}$, so there is $x \in C$ with $\pi[x] \in N$. So $\pi[x] = \pi(y)$ for some $y \in M$. However, $y \subseteq M$ by its size and so $\pi(y) = \pi[y] = \pi[x]$, ergo $x = y \in M \cap C$.

The other proofs work similarly. For (4) we note $\pi((x_i)_{i<j}) = (\pi(x_i))_{i<j}$. □

In [Kru09, Theorem 6.5], Krueger shows the following theorem:

Theorem 7.1.2. *Assume μ is a regular uncountable cardinal and $2^\mu = \mu^+$. The following are equivalent:*

1. *There is a disjoint stationary sequence on μ^+.*
2. *There are stationarily many $N \in [H(\mu^+)]^\mu$ which are internally unbounded but not internally club.*

There is the following additional result: This was shown by Cox in [Cox21], where he states it as being folklore:

Theorem 7.1.3. *Assume μ is a regular uncountable cardinal and $2^\mu = \mu^+$. The following are equivalent:*

1. *The approachability property fails at μ, i.e. $\mu^+ \notin I[\mu^+]$.*
2. *There are stationarily many $N \in [H(\mu^+)]^\mu$ which are internally unbounded but not internally approachable.*

We will prove both results here and show later that the assumption $2^\mu = \mu^+$ is necessary to obtain them.

We start with a preliminary lemma:

Lemma 7.1.4. *Assume μ is a regular uncountable cardinal and $2^\mu = \mu^+$. There exists a club $C \subseteq [H(\mu^+)]^\mu$ such that whenever $N \in C$ and $N \cap \mu^+$ is an ordinal with cofinality μ, N is internally unbounded.*

Proof. Let $(N_\alpha)_{\alpha\in\mu^+}$ be an sequence of elements of $[H(\mu^+)]^\mu$ such that $(N_\alpha)_{\alpha\leq\beta} \in N_{\beta+1}$ and $N_\beta = \bigcup_{\alpha<\beta} N_\alpha$ for limit β. By using an appropriate bookkeeping we can make sure that $\{N_\alpha \mid \alpha \in \mu^+\}$ is unbounded in $[H(\mu^+)]^\mu$. It is clearly closed. Let C be the club of all N_α such that $N_\alpha \cap \mu^+ = \alpha$. If $N_\alpha \in C$ and $N_\alpha \cap \mu^+ = \alpha$ has cofinality μ, $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$. If $x \subseteq N_\alpha$ has size $<\mu$, $x \subseteq N_\beta$ for some $\beta < \alpha$ and clearly $N_\beta \in N_{\beta+1} \subseteq N_\alpha$. □

We can now prove both theorems using very similar arguments:

Proof of Theorem 7.1.2. Let $(\mathcal{S}_\alpha)_{\alpha\in S}$ be a disjoint stationary sequence, i.e. $S \subseteq \mu^+ \cap \mathrm{cof}(\mu)$ is stationary, $\mathcal{S}_\alpha$ is stationary in $[\alpha]^{<\mu}$ for every $\alpha \in S$ and the $\mathcal{S}_\alpha$ are pairwise disjoint.

Let $M \prec (H(\mu^+), (\mathcal{S}_\alpha)_{\alpha\in S}, \in)$ be such that $\alpha := M \cap \mu^+ \in S$ and $M \in C$ (where C is the club from Lemma 7.1.4). Then M is internally unbounded.

Claim. *M is not internally club.*

Proof. Assume toward a contradiction that M is internally club. In particular, the set $\{x \cap \mu^+ \mid x \in [M]^{<\mu} \cap M\}$ contains a club in $[\alpha]^{<\mu}$. Ergo there is $x \in [M]^{<\mu} \cap M \cap \mathcal{S}_\alpha$. By elementarity, the unique β with $x \in \mathcal{S}_\beta$ is in M. But $\beta = \alpha$ which is not in M, a contradiction. □

Now assume there exist stationarily many $N \in [H(\mu^+)]^\mu$ which are internally unbounded but not internally club. Let S be the set of $N \cap \mu^+$ where $N \in C$ (again the club from Lemma 7.1.4) is internally unbounded but not internally club and $N \prec (H(\mu^+), F, \in)$ where $F\colon \mu^+ \to H(\mu^+)$ is a bijection. Then any $\alpha \in S$ has cofinality μ (otherwise N could not be internally unbounded because it could not cover a cofinal subset of $N \cap \mu^+$ of ordertype $<\mu$). Also S is stationary by assumption. Let $\alpha \in S$, $\alpha = N \cap \mu^+$. Because N is not internally club by assumption (which implies that $[N\cap\mu^+]^{<\mu} \cap N$ does not contain a club in $[N\cap\mu^+]^{<\mu}$ since $F\colon \mu^+ \to H(\mu^+)$ is bijective) we can let $\mathcal{S}_\alpha$ be the least (according to some well-order on $H(\mu^+)$ which is definable in $(H(\mu^+), F)$ because $|H(\mu^+)| = \mu^+$) stationary subset of $[\alpha]^{<\mu}$ which has size μ and is disjoint from N.

We claim that $(\mathcal{S}_\alpha)_{\alpha\in S}$ is a disjoint stationary sequence. Let $\alpha, \beta \in S$, $\alpha < \beta$. Let N_α, N_β be such that $\alpha = N_\alpha \cap \mu^+$, $\beta = N_\beta \cap \mu^+$. Then $N_\alpha \in N_\beta$ (because C is an $\in$-chain), so $\mathcal{S}_\alpha$ (which is definable from N_α) is in N_β as well. As $|\mathcal{S}_\alpha| = \mu$, $\mathcal{S}_\alpha \subseteq N_\beta$ and $\mathcal{S}_\alpha$ is disjoint from $\mathcal{S}_\beta$. □

A similar argument shows Theorem 7.1.3:

Proof of Theorem 7.1.3. Assume $\mu^+ \in I[\mu^+]$, witnessed by $\overline{a} = (a_\alpha)_{\alpha<\mu^+}$, i.e. there is a club D such that for any $\gamma \in D$ there is $A \subseteq \gamma$ unbounded in γ with $\mathrm{otp}(A) = \mathrm{cf}(\gamma)$ such that $\{A \cap \beta \mid \beta < \gamma\} \subseteq \{a_\beta \mid \beta < \gamma\}$.

Now assume toward a contradiction that there are stationarily many $N \in [H(\mu^+)]^\mu$ which are internally unbounded but not internally approachable. In particular, there is such an N with the following additional properties:

1. $N \in C$ (again the club from Lemma 7.1.4)
2. $\mathrm{cf}(N \cap \mu^+) = \mu$
3. $N \prec (H(\mu^+), F, \overline{a}, \in)$, where $F\colon \mu^+ \to H(\mu^+)$ is a bijection.
4. $N \cap \mu^+ \in D$.

So there is $A \subseteq \gamma$ with $\mathrm{otp}(A) = \mu$ such that $\{A \cap \beta \mid \beta < \gamma\} \subseteq \{a_\beta \mid \beta < \gamma\}$. In particular, for any $\beta < \gamma$, $A \cap \beta \in N$ and so is $(A \cap \delta)_{\delta\in\beta}$ (as this sequence is definable from $A \cap \beta$). However, this implies that N is internally approachable: By elementarity, $F[\gamma] = N$ and so $N = \bigcup_{\alpha\in A} F[\alpha]$ but for any $\beta < \gamma$, $(F[\alpha])_{\alpha\in A\cap\beta} \in N$. Let $f\colon \{i+1 \mid \alpha \in \mu\} \to A$ be order-preserving and let $f(\gamma) := \bigcup_{i<\gamma} f(i)$ for limit α. Let $x_i := F[f(i)]$. Then $(x_i)_{i<\mu}$ is an increasing and continuous sequence with union N. Moreover, for any $j < \mu$, $(x_i)_{i<j}$ is definable from $A \cap f(j)$ and F, so it is in N.

Now assume that $\mu^+ \notin I[\mu^+]$. By a result of Shelah (which states $\mu^+ \cap \mathrm{cof}(<\mu) \in I[\mu^+]$; see [She91], Lemma 4.4), this means that $\mu^+ \cap \mathrm{cof}(\mu) \notin I[\mu^+]$. As $2^\mu = \mu^+$, we can fix an enumeration $\overline{a} = (a_\alpha)_{\alpha\in\mu^+}$ of $[\mu^+]^{<\mu}$ and since $\mu^+ \cap \mathrm{cof}(\mu) \notin I[\mu^+]$, there is a stationary set $S \subseteq \mu^+ \cap \mathrm{cof}(\mu)$ such that for every $\gamma \in S$ and every unbounded $A \subseteq \gamma$ there is some β such that $A \cap \beta \notin \{a_\alpha \mid \alpha < \gamma\}$.

Now let $D \subseteq [H(\mu^+)]^\mu$ be club. Let $N \in D$ have the following properties:

1. $N \in C$ (the club from Lemma 7.1.4)
2. $N \prec (H(\mu^+), \overline{a}, \in)$
3. $\gamma := N \cap \mu^+ \in S$

In particular N is internally unbounded. Assume to the contrary that N is internally approachable. Then there exists a continuous sequence $(x_\alpha)_{\alpha<\mu}$ of elements of $[N \cap \mu^+]^{<\mu}$ with union $N \cap \mu^+$ such that $(x_\alpha)_{\alpha<\beta} \in N$ for all $\beta \in \mu$. By elementarity, $\{a_\alpha \mid \alpha < \gamma\}$ equals $[\mu^+]^{<\mu} \cap N$. Let $A := \{\sup(x_\alpha) \mid \alpha < \mu\}$. Clearly A is unbounded in γ and has ordertype μ. Additionally, given any $\beta < \gamma$, $A \cap \beta = \{\sup(x_\alpha) \mid \alpha < \delta\}$ for some $\delta < \mu$ and so $A \cap \beta \in [\mu^+]^{<\mu} \cap N = \{a_\alpha \mid \alpha < \gamma\}$. This is of course a contradiction. $\square$

The previous two theorems combine easily to show that under $2^\mu = \mu^+$, the existence of a disjoint stationary sequence implies the failure of the approachability property. However, this does not depend on $2^\mu = \mu^+$, as shown by Krueger:

Theorem 7.1.5 ([Kru09], Corollary 3.7)**.** *Suppose μ is a regular uncountable cardial and $(\mathcal{S}_\alpha)_{\alpha\in S}$ is a disjoint stationary sequence on μ^+. Then no stationary subset of S is in $I[\mu^+]$. In particular, the approachability property fails at μ.*

We will later give a different proof. Let us note that the reverse direction is false in general, i.e. it is consistent that there is no disjoint stationary sequence on μ^+ but the approachability property still fails at μ: This was shown by Cox in [Cox21] who produced a model of PFA where there is no disjoint stationary sequence. Levine in [Lev24] produced a similar model from a Mahlo cardinal where $\neg\,\mathrm{AP}_{\omega_1} \wedge \neg\,\mathrm{DSS}(\omega_2)$ holds. We will prove in section 9 that this is consistent at larger cardinals by showing that for any supercompact λ and any regular $\mu < \lambda$ there is a forcing extension where AP_μ and $\mathrm{DSS}(\mu^+)$ both fail (the failure of AP_μ will even be a consequence of $\mathrm{ISP}(\omega_1, \mu^+, \geq \mu^+)$).

As is clear from the proofs, the requirement that $2^\mu = \mu^+$ is used to obtain a bijection $F\colon \mu^+ \to H(\mu^+)$ in order to relate the properties $\mathrm{DSS}(\mu^+)$ and AP_μ, which are concerned with subsets of μ^+, to properties depending on subsets of $H(\mu^+)$. Due to this, we define the following:

Definition 7.1.6. Let $N \in [H(\Theta)]^\mu$.

1. N is On-*internally unbounded* if $[N \cap \Theta]^{<\mu} \cap N$ is unbounded in $[N \cap \Theta]^{<\mu}$.
2. N is On-*internally stationary* if $[N \cap \Theta]^{<\mu} \cap N$ is stationary in $[N \cap \Theta]^{<\mu}$.
3. N is On-*internally club* if $[N \cap \Theta]^{<\mu} \cap N$ contains a club in $[N \cap \Theta]^{<\mu}$.
4. N is On-*internally approachable* if there is a sequence $(x_i)_{i<\mu}$ of elements of $[N \cap \Theta]^{<\mu}$ such that $\bigcup_{i<\mu} x_i = N \cap \Theta$ and $(x_i)_{i<j} \in N$ for every $j < \mu$.

Clearly, if $F\colon \Theta \to H(\Theta)$ is a bijection and $N \prec (H(\Theta), F, \in)$, N is On-internally unbounded (stationary; club; approachable) if and only if N is internally unbounded (stationary; club; approachable) since N is closed under the function $x \mapsto F[x]$ and $F\colon N \cap \Theta \to N$ is a bijection by elementarity.

By carefully studying the proofs of the previous theorems and noting that any N with $\mathrm{cf}(N \cap \mu^+) = \mu$ is internally unbounded, we can see that by moving to the On-versions of the internal structure, we can remove the reliance on $2^\mu = \mu^+$. For disjoint stationary sequences, we have the following:

Theorem 7.1.7. *Let μ be a regular uncountable cardinal. The following are equivalent:*

1. *There exists a disjoint stationary sequence on μ^+.*
2. *There are stationarily many $N \in [H(\mu^+)]^\mu$ which are* On-*internally unbounded but not* On-*internally club.*

In the proof of Theorem 7.1.3, we also used the assumption $(\mu^+)^{<\mu} = \mu^+$ to be able to find a sequence $(x_\alpha)_{\alpha<\mu^+}$ enumerating all of $[\mu^+]^{<\mu}$. Therefore the "improved" theorem still needs that assumption.

Theorem 7.1.8. *Let μ be a regular uncountable cardinal. Define the following statements:*

1. $\mu^+ \notin I[\mu^+]$.
2. *There are stationarily many $N \in [H(\mu^+)]^\mu$ which are* On*-internally unbounded but not* On*-internally approachable.*

Then (2) implies (1). Furthermore, if $(\mu^+)^{<\mu} = \mu^+$, (1) also implies (2).

This implies Theorem 7.1.5 almost immediately:

Proof of Theorem 7.1.5. Suppose μ is a regular uncountable cardinal and $(\mathcal{S}_\alpha)_{\alpha \in S}$ is a disjoint stationary sequence on μ^+. Then it follows that there is a club $C \subseteq [H(\mu^+)]^\mu$ such that any $M \in C$ with $M \cap \mu^+ \in S$ is On-internally unbounded but not On-internally club.

Now assume toward a contradiction that there is a stationary set $A \subseteq S$ and a sequence $(x_i)_{i \in \mu^+}$ of elements of $[\mu^+]^{<\mu}$ such that any ordinal in A is approachable with respect to $(x_i)_{i \in \mu^+}$. It follows that there is a structure $M \in C$ which is elementary in $(H(\mu^+), \in, \overline{x})$ such that $M \cap \mu^+ \in A$. However, since $A \subseteq S$, M is not On-internally club. Because $M \cap \mu^+$ is approachable with respect to $\overline{x}$, there exists $B \subseteq M \cap \mu^+$ unbounded with minimal ordertype such that $\{B \cap \beta \mid \beta < M \cap \mu^+\} \subseteq \{x_\beta \mid \beta < M \cap \mu^+\}$. Furthermore, any x_β is in M. Since $\{B \cap \beta \mid \beta < M \cap \mu^+\}$ is club in $[M \cap \mu^+]^{<\mu}$, M is On-internally club, a contradiction. □

We will later see *en passant* that the On-versions of internal approachability are in general not equivalent to the non-On-versions.

7.2 Distinguishing Internal Stationarity and Clubness

Levine partially answered question (1) in [Lev24] by using a modified version of Mitchell forcing to construct a model where $\mathrm{DSS}(\omega_2) \wedge \mathrm{DSS}(\omega_3)$ holds. We will define a "mixed-support product" of instances of this forcing to obtain a model where $\mathrm{DSS}(\omega_{n+2})$ holds for every $n \in \omega$, thus fully answering question (1).

In the same paper, Levine noticed that in the model constructed to have disjoint stationary sequences on both ω_2 and ω_3, the notions of internal stationarity and clubness are distinct for $[H(\omega_2)]^{\omega_1}$ and $[H(\omega_3)]^{\omega_2}$, partially answering question (2). The same (and even more) is true in our case, i.e. for any $n \in \omega$, the notions of internal stationarity and clubness are distinct for $[H(\Theta)]^{\omega_{n+1}}$ (with $\Theta > \omega_{n+1}$ arbitrary), thus fully answering question (2).

7.2.1 A Single Cardinal

We define the basic forcing we want to use to distinguish internal stationarity and internal clubness:

Definition 7.2.1. Let $\mu < \kappa$ be regular cardinals such that κ is inaccessible and μ is uncountable. Let $A := \kappa$ and define F by induction. If $\gamma = \delta + 1$, where δ is inaccessible, let $F(\gamma)$ be an $\mathbb{M}(\omega, \mu, \kappa, A, F, \gamma)$-name for $\dot{\mathrm{Coll}}(\check{\mu}, \check{\delta})$. Otherwise, let $F(\gamma) := \check{1}$.

Define $\mathbb{M}_0(\mu, \gamma) := \mathbb{M}(\omega, \mu, \kappa, A, F, \gamma)$. If $\nu < \kappa$ and G is any $\mathbb{M}_0(\mu, \nu)$-generic filter, define (in $V[G]$) $\mathbb{M}_0(G, \mu, \kappa \smallsetminus \nu) := \mathbb{M}(G, \omega, \mu, \kappa \smallsetminus \nu, A, F)$. Let $\mathbb{T}_0(\mu, \kappa)$ and $\mathbb{T}_0(G, \mu, \kappa \smallsetminus \nu)$ be defined accordingly (as in Remarks 6.1.3 and 6.2.4).

The poset $\mathbb{M}_0(\mu, \kappa)$ is very similar to the original Mitchell Forcing defined by Mitchell in [Mit72] and was employed by Levine in [Lev24] to construct disjoint stationary sequences on two successive cardinals. In [Jak23], we used a forcing which can be seen as a "mixed support iteration" of the above poset (in a more general sense). However, while working on later papers, we noticed that a "mixed-support product" was sufficient to obtain the desired results and easier to work with.

As in the papers by Krueger and Levine, we need that forcing with $\mathrm{Add}(\omega)$ adds stationarily many new sets. The following fact was shown by Krueger and refines a theorem of Gitik:

Fact 7.2.2 ([Kru09], Theorem 7.1)**.** *Suppose $V \subseteq W$ are models of set theory with the same ordinals, $W \smallsetminus V$ contains a real, μ is a regular uncountable cardinal in W, $X \in V$ is such that $(\mu^+)^W \subseteq X$ and Θ is regular in W with $X \subseteq H(\Theta)$. Then in W the set $\{N \in [H^W(\Theta)]^{<\mu} \cap \mathrm{IA}(\omega) \mid N \cap X \notin V\}$ is stationary in $[H^W(\Theta)]^{<\mu}$.*

And we can immediately show the $\mathbb{M}_0$ forces the distinction for a single cardinal.

Theorem 7.2.3. *Let κ be a Mahlo cardinal and $\mu < \kappa$ regular. Let G be $\mathbb{M}_0(\mu, \kappa)$-generic. In $V[G]$, for every $\Theta \geq \kappa$ there are stationarily many $N \in [H^{V[G]}(\Theta)]^{\mu}$ which are internally stationary but not internally club.*

Proof. In $V[G]$, let $F \colon [H^{V[G]}(\Theta)]^{<\omega} \to [H^{V[G]}(\Theta)]^{\mu}$. Let $\dot{F}$ be a name for F. Our aim is to find $Z \in [H^{V[G]}(\Theta)]^{\mu}$ which is closed under F and internally stationary but not internally club.

To this end, fix $\Theta' > \Theta$ such that $H^V(\Theta')$ contains all the relevant objects. Applying the Mahloness of κ in V (see Theorem 5.2.1), find $M \prec H^V(\Theta')$ with the following:

1. $\nu := |M| = M \cap \kappa \in \kappa$ is inaccessible
2. $\dot{F}, \mu, \kappa, \Theta, \mathbb{M}_0(\mu, \kappa) \in M$
3. ${}^{<\nu}M \subseteq M$.

Let $\pi \colon M \to N$ be the Mostowski-Collapse of M. By Lemma 5.3.6, in $V[G]$ the collapse π extends to $\pi \colon M[G] \to N[G']$, where $G' := G \cap \mathbb{M}_0(\mu, \nu)$. As $\dot{F}^G \in M[G]$, $M[G] \cap H^{V[G]}(\Theta)$ is closed under $\dot{F}^G$. Furthermore, by Lemma 5.3.5, $N[G']$ is closed under $<\nu$-sequences in $V[G']$.

Our aim is to show that $M[G] \cap H^{V[G]}(\Theta)$ is internally stationary but not internally club in $V[G]$.

Claim. *$M[G]$ is internally stationary in $V[G]$.*

Proof. By Lemma 7.1.1, it suffices to show that $N[G']$ is internally stationary. However, since $[N[G']]^{<\mu} \cap V[G'] \subseteq N[G']$, this is clear by Lemma 2.4.6: $V[G]$ is an extension of $V[G']$ by a poset which is the projection of the product of a c.c.c. and a μ-strategically closed poset (see Lemma 6.2.6). Since c.c.c. posets do not destroy the stationarity of subsets of $[X]^{<\mu}$, by Lemma 2.4.6 $[N[G']]^{<\mu} \cap V[G']$ is stationary in $[N[G']]^{<\mu}$ in $V[G]$. However, the former set is equal to $[N[G']]^{<\mu} \cap N[G']$. □

So, since $\Theta \in M[G]$ which implies $H^{V[G]}(\Theta) \in M[G]$ by elementarity, $M[G] \cap H^{V[G]}(\Theta)$ is internally stationary: Let $c \subseteq [M[G] \cap H^{V[G]}(\Theta)]^{<\mu}$ be club. Then

$$c' := \{m \in [M[G]]^{<\mu} \mid m \cap H^{V[G]}(\Theta) \in c\}$$

is club in $[M[G]]^{<\mu}$. Therefore there is $m \in c' \cap M[G]$. By its size $m \cap H^{V[G]}(\Theta) \in H^{V[G]}(\Theta)$ and by elementarity $m \cap H^{V[G]}(\Theta) \in M[G]$. Ergo $m \cap H^{V[G]}(\Theta) \in c \cap (M[G] \cap H^{V[G]}(\Theta))$.

Claim. *$M[G] \cap H^{V[G]}(\Theta)$ is not internally club in $V[G]$.*

Proof. Assume otherwise. Then in particular, since $\nu \subseteq M[G] \cap H^{V[G]}(\Theta)$, there is $c \subseteq [\nu]^{<\mu}$ club such that $c \subseteq M[G]$: Let $d \subseteq M[G] \cap H^{V[G]}(\Theta)$ be club in $[M[G] \cap H^{V[G]}(\Theta)]^{<\mu}$. Then $\{m \cap \kappa \mid m \in d\}$ contains a club in $[M[G] \cap H^{V[G]}(\Theta) \cap \kappa]^{<\mu} = [\nu]^{<\mu}$ and is contained in $M[G]$ by elementarity.

Furthermore, $c \subseteq N[G']$, since the elements of c are not moved by π and lastly, $c \subseteq V[G']$, so in summary (assuming the claim fails), there exists a club $c \subseteq [\nu]^{<\mu}$ which is a subset of $V[G']$. We will show that this is not the case.

Let $G'' := G \cap \mathbb{M}_0(\mu, \nu+1)$. Since $\mathbb{M}_0(\mu, \nu+1) \cong \mathbb{M}_0(\mu, \nu) * \mathrm{Add}(\omega)$, which is ν-c.c., we can apply Fact 7.2.2 with $V[G']$ in lieu of V, $V[G'']$ in lieu of W, ν in lieu of X and μ starring as itself (noting that $(\mu^+)^{V[G'']} = \nu$) to see that in $V[G'']$, the set

$$S' := \{Y \in [H^{V[G'']}(\Theta)]^{<\mu} \cap \mathrm{IA}(\omega) \mid Y \cap \nu \notin V[G']\}$$

is stationary in $[H^{V[G'']}(\Theta)]^{<\mu}$. Since $V[G]$ is an extension of $V[G'']$ by $\mathbb{M}_0(\mu, \kappa \smallsetminus (\nu+1))$ which can be projected onto by the product of a μ-strategically closed and a c.c.c. poset, S' is still stationary in $[H^{V[G'']}(\Theta)]^{<\mu}$ in $V[G]$ by Lemma 2.4.5. This clearly implies that $\{Y \cap \nu \mid Y \in S'\}$, which is disjoint from $V[G']$, is stationary in $[\nu]^{<\mu}$, a contradiction. □

So $M[G] \cap H^{V[G]}(\Theta)$ is a set closed under $\dot{F}^G$ which is internally stationary but not internally club. □

By Theorem 7.1.2:

Theorem 7.2.4. *$\mathbb{M}_0(\mu, \kappa)$ forces that there exists a disjoint stationary sequence on μ^+.*

Proof. $\mathbb{M}_0(\mu, \kappa)$ forces $2^\mu = \kappa = \mu^+$. Under this assumption, the existence of a disjoint stationary sequence on μ^+ and the existence of stationarily many $N \in [H(\mu^+)]^\mu$ which are internally unbounded but not internally club are equivalent. □

Remark 7.2.5. We can also straightforwardly define the disjoint stationary sequence: Let S consist of all inaccessible cardinals below κ and, given $\nu \in S$, let

$$S_\nu := [\nu]^{<\mu} \cap (V[G \cap \mathbb{M}(\mu, \nu+1)] \smallsetminus V[G \cap \mathbb{M}(\mu, \nu)])$$

Clearly S remains a stationary set and becomes a subset of $\mu^+ \cap \mathrm{cof}(\mu)$. Furthermore, the S_ν are of course disjoint. The arguments in the proof of Theorem 7.2.3 show that each S_ν is stationary in $[\nu]^{<\mu}$.

7.2.2 Infinitely Many Cardinals

We now extend the previous construction to infinitely many cardinals. For the rest of this section we fix an increasing sequence $(\kappa_n)_{n\in\omega}$ of Mahlo cardinals. Also let $\kappa_{-1} := \omega_1$. Assume without loss of generality that $2^{\sup_n \kappa_n} = (\sup_n \kappa_n)^+$. We want to force with the product $\prod_n \mathbb{M}_0(\kappa_{n-1}, \kappa_n)$. However, there is the following caveat: If we were to take the product with finite support, we would not have a sufficiently closed term ordering on the tail forcing. If we were to take the product with infinite (i.e. full) support, we would have too large antichains in the base ordering on the tail forcing. Due to this, we take the "mixed-support" product which we will now define. Recall that we view $\mathbb{M}_0(\mu, \kappa)$ as an order on the product $\mathrm{Add}(\omega, \kappa) \times \mathbb{T}_0(\mu, \kappa)$.

Definition 7.2.6. The poset $\mathbb{I}$ consists of pairs (p, q) such that

1. p is a function on ω such that for $n \in \omega$, $p(n) \in \mathrm{Add}(\omega, \kappa_n)$ and $p(n) = \emptyset$ for all but finitely many n.
2. q is a function on ω such that for $n \in \omega$, $q(n) \in \mathbb{T}_0(\kappa_{n-1}, \kappa_n)$

We let $(p', q') \leq (p, q)$ if and only if for all $n \in \omega$, $(p'(n), q'(n)) \leq (p(n), q(n))$ in $\mathbb{M}_0(\kappa_{n-1}, \kappa_n)$. For a natural number n, we let $\mathbb{I}_n$ consist of those $(p, q) \in \mathbb{I}$ such that $p(k)$ and $q(k)$ are trivial for $k \geq n$ and we let $\mathbb{I}^n$ consist of those $(p, q) \in \mathbb{I}$ such that $p(k)$ and $q(k)$ are trivial for $k < n$.

Remark 7.2.7. We note the following:

1. For a given n, $\mathbb{I}$ is isomorphic to $\mathbb{I}_n \times \mathbb{I}^n$. Furthermore, $\mathbb{I}_n$ is isomorphic to the "normal" product $\prod_{k<n} \mathbb{M}_0(\kappa_{k-1}, \kappa_k)$ (which is κ_{n-1}-Knaster).
2. We can view $\mathbb{I}$, $\mathbb{I}_n$ and $\mathbb{I}^n$ as orderings on products. For $\mathbb{I}$ and $\mathbb{I}^n$, the base ordering is isomorphic to $\mathrm{Add}(\omega, \sup_n \kappa_n)$. For $\mathbb{I}^n$, the term ordering is $< \kappa_{n-1}$-closed. It follows that $\mathbb{I}^n$ can be projected onto from the product of an ω_1-Knaster and a $< \kappa_{n-1}$-closed poset (which we denote by $\mathbb{T}^n$).

Before showing that $\mathbb{I}$ forces the distinction for $[H(\Theta)]^{\omega_{n+1}}$ for every n, we need a definition and two preservation lemmas. In general, we do not expect the distinction to be preserved under forcings which increase certain powerset sizes (we will take advantage of a similar situation in the last section of this chapter). However, $\mathbb{M}_0(\mu, \kappa)$ forces the distinction to hold in a very strong way:

Definition 7.2.8. Let μ be a regular cardinal.

1. Let $\kappa \leq \Theta$ be cardinals and $N \prec H(\Theta)$ with size μ. We say that N is *κ-internally club* if $[N \cap \kappa]^{<\mu} \cap N$ contains a club in $[N \cap \kappa]^{<\mu}$.
2. We let $\mathrm{GDSC}(\mu^+)$ (*G*lobal *D*istinction Between Internal *S*tationarity and *C*lubness) state that for any $\Theta \geq \mu^+$ there are stationarily many $N \in [H(\Theta)]^\mu$ which are internally stationary but not μ^+-internally club.

Clearly, if N is internally club and $\kappa \in N$, N is κ-internally club (since N is closed under intersections). Ergo $\mathrm{GDSC}(\mu^+)$ implies that for any $\Theta \geq \mu^+$ there are stationarily many $N \in [H(\Theta)]^\mu$ which are internally stationary but not internally club. Our proof of Theorem 7.2.3 shows the following stronger version:

Corollary 7.2.9. *Let κ be a Mahlo cardinal and $\mu < \kappa$ regular. Let G be $\mathbb{M}_0(\mu, \kappa)$-generic. In $V[G]$, $\mathrm{GDSC}(\mu^+)$ holds.*

We also remark the following:

Remark 7.2.10. To show $\mathrm{GDSC}(\mu^+)$ it suffices to prove that for the statement holds for sufficiently large Θ: If $C \subseteq [H(\Theta)]^\mu$ is club and $\Theta' > \Theta$, then

$$C' := \{M \in [H(\Theta')]^\mu \mid M \cap H(\Theta) \in C \wedge \Theta \in M\}$$

is club in $[H(\Theta')]^\mu$. If $M \in C'$ is internally stationary but not μ^+-internally club, the same is true for $M \cap H(\Theta)$: Let $c \subseteq [M \cap H(\Theta)]^{<\mu}$ be club. Then $c' := \{l \in [M]^{<\mu} \mid l \cap H(\Theta) \in c\}$ is club in $[M]^{<\mu}$, so there is $l \in c' \cap M$. Then $l \cap H(\Theta) \in c \cap (M \cap H(\Theta))$. That $M \cap H(\Theta)$ is not μ^+-internally club is clear.

This stronger property is more easily preserved.

Lemma 7.2.11. *Let μ be a regular cardinal. Assume $\mathrm{GDSC}(\mu^+)$ holds and $\mathbb{P}$ is a μ-c.c. forcing. Then $\mathbb{P}$ forces $\mathrm{GDSC}(\mu^+)$.*

Remark 7.2.12. Note that it is not enough that c.c.c. forcings preserve stationarity and thus not containing a club since in most cases, $H^{V[G]}(\Theta) \neq H^V(\Theta)$. Additionally, since it is possible that $H^{V[G]}(\Theta) \neq H^V(\Theta)[G]$, we do not expect a level-by-level preservation.

Proof. Let G be $\mathbb{P}$-generic. In $V[G]$, let $\Theta > \mu$ and let $F\colon [H^{V[G]}(\Theta)]^{<\omega} \to [H^{V[G]}(\Theta)]^\mu$. Let $\dot{F}$ be a name for F. In V let Θ' be large and $N \prec H^V(\Theta')$ of size μ such that $\dot{F}$, Θ, $\mathbb{P}$, μ^+ are in N, $\mu \subseteq N$ and N is internally stationary but not μ^+-internally club. Let G be $\mathbb{P}$-generic. We want to show that $N[G]$ is internally stationary but not μ^+-internally club (then clearly $N[G] \cap H^{V[G]}(\Theta)$ is internally stationary but not μ^+-internally club and closed under $\dot{F}^G$). We note that $N[G] \cap V = N$ by the μ-c.c. of $\mathbb{P}$.

Claim. *$N[G]$ is internally stationary in $V[G]$.*

Proof. Let $c \subseteq [N[G]]^{<\mu}$ be club in $[N[G]]^{<\mu}$. Let (still in $V[G]$) $d \subseteq [N]^{<\mu}$ consist of all $m \in [N]^{<\mu}$ such that $m \prec N$ and $m \cap \mu \in \mu$. Clearly d is club in $[N]^{<\mu}$ (in $V[G]$). Ergo $\tilde{d} := \{m[G] \mid m \in d\}$ is club in $[N[G]]^{<\mu}$, so we can assume that $c \subseteq \tilde{d}$. Define $c' := \{m \cap V \mid m \in c\}$. Then c' contains a club in $[N[G] \cap V]^{<\mu} = [N]^{<\mu}$. Since $\mathbb{P}$ is μ-c.c. and N is internally stationary in V it is internally stationary in $V[G]$ and so there is $m' \in c' \cap N$. It follows that $m' = m \cap V$ for $m \in c$ and $m = n[G]$ for $n \in d$. Since $n \cap \mu \in \mu$ and $\mathbb{P}$ is μ-c.c. we have

$$m' = m \cap V = n[G] \cap V = n$$

which shows that $n \in N$. Ergo $m = n[G]$, which is definable from n and G, is in $N[G] \cap c$. $\square$

Claim. *$N[G]$ is not μ^+-internally club in $V[G]$.*

Proof. Assume toward a contradiction that $c \subseteq N[G]$ is club in $[N[G] \cap \mu^+]^{<\mu} = [N \cap \mu^+]^{<\mu}$. By the μ-c.c. of $\mathbb{P}$ there is $c' \subseteq c$ such that $c' \in V$ and c' is club in $[N \cap \mu^+]^{<\mu}$. However, this implies $c' \subseteq N[G] \cap V = N$, a contradiction as N is not μ^+-internally club in V. $\square$

In summary we have produced a set closed under $\dot{F}^G$ which is internally stationary but not μ^+-internally club, finishing the proof. $\square$

We also have a downward preservation. As before, since we cannot assume $H^{V[G]}(\Theta) = H^V(\Theta)[G]$, we work with GDSC.

Lemma 7.2.13. *Assume W is a forcing extension of V by a forcing order $\mathbb{P}$ which is $<\mu^+$-distributive for some regular μ. Assume that* $\mathrm{GDSC}(\mu^+)$ *holds in W. Then* $\mathrm{GDSC}(\mu^+)$ *holds in V.*

Remark 7.2.14. The important point is that we require the strongest form of internal stationarity (it holding at the highest level of the model) and also the failure of the weakest form of internal clubness (at the lowest level of the model) in order to make the proof work.

Proof. Let G be $\mathbb{P}$-generic with $W = V[G]$. We note that by the distributivity $(\mu^+)^W = (\mu^+)^V$. In V, let Θ be so large that $\mathbb{P} \in \Theta$ and let $C \subseteq [H^V(\Theta)]^\mu$ be club. If suffices to show that C contains an element which is internally stationary but not μ^+-internally club.

In W, C is still club in $[H^V(\Theta)]^\mu$ by the distributivity of $\mathbb{P}$. We first prove the following statement reminiscent of properness:

Claim. *In W, the set*

$$D := \{M \in [H^V(\Theta)]^\mu \mid M[G] \cap V = M\}$$

is club in $[H^V(\Theta)]^\mu$.

Proof. For closure, notice (for $\delta \leq \mu$):

$$\left(\bigcup_{i<\delta} M_i\right)[G] \cap V = \left(\bigcup_{i<\delta} M_i[G]\right) \cap V = \bigcup_{i<\delta}(M_i[G] \cap V)$$

For unboundedness, let $M_0 \in [H^V(\Theta)]^\mu$ be arbitrary. Inductively define M_{n+1} as the union of M_n and $(M_n[G] \cap V)$. Then

$$\left(\bigcup_{n\in\omega} M_n\right)[G] \cap V = \bigcup_{n\in\omega} M_n$$

since, given some $\tau \in M_n$ with $\tau^G \in V$, $\tau^G \in M_n[G] \cap V = M_{n+1}$. $\square$

By the size of Θ we have $H^{V[G]}(\Theta) = H^V(\Theta)[G]$, so $E' := \{M[G] \mid M \in D \cap C\}$ is club in $[H^{V[G]}(\Theta)]^\mu$. In summary, there is $M \in D \cap C$ such that $M[G] \prec (H^{V[G]}(\Theta), H^V(\Theta), \in)$ is internally stationary but not μ^+-internally club in $V[G]$. We want to show that M is internally stationary but not μ^+-internally club in V.

Claim. *M is internally stationary in V.*

Proof. Let $c \subseteq [M]^{<\mu}$ be club in V. It is also club in $V[G]$ by the distributivity. So $\{m[G] \mid m \in c\}$ is club in $[M[G]]^{<\mu}$ in $V[G]$ so there is $m \in c$ with $m[G] \in M[G]$ such that $m[G] \cap V = m$ (as in the first claim). Since $M[G] \prec (H^{V[G]}(\Theta), H^V(\Theta), \in)$, we know $m[G] \cap V = m[G] \cap H^V(\Theta) \in M[G]$ and by the distributivity of $\mathbb{P}$, $m[G] \cap V \in M[G] \cap V = M$. So $m[G] \cap V = m \in c \cap M$. $\square$

Claim. *M is not μ^+-internally club in V.*

Proof. This is clear since $M \cap \mu^+ = M[G] \cap \mu^+$ and $M \subseteq M[G]$. If M were μ^+-internally club in V, $M[G]$ would be μ^+-internally club in W, a contradiction. $\square$

As before, we have produced an element of C which is internally stationary but not μ^+-internally club. $\square$

Now we can prove the main result:

Theorem 7.2.15. *Let G be $\mathbb{I}$-generic. In $V[G]$, $\mathrm{GDSC}(\omega_{n+2})$ holds for every $n \in \omega$.*

So in particular, for any n and $\Theta > \omega_{n+1}$ there are stationarily many $N \in [H^{V[G]}(\Theta)]^{\omega_{n+1}}$ which are internally stationary but not internally club.

Proof. Notice first of all that $\mathbb{I}$ forces $\kappa_n = \omega_{n+2}$ for every n: For any given n, forcing with $\mathbb{M}_0(\kappa_{n-1}, \kappa_n)$ forces $\kappa_n = \kappa_{n-1}^+$. By induction, forcing with $\mathbb{I}_{n+1}$ forces $\kappa_n = \omega_{n+2}$. Since we can project onto $\mathbb{I}^{n+1}$ from the product of a $<\kappa_n$-closed and a c.c.c. poset, $\mathbb{I}$ also forces $\kappa_n = \omega_{n+2}$ for any $n \in \omega$.

We now prove that $\mathbb{I}$ forces $\mathrm{GDSC}(\omega_{n+2})$: Write $\mathbb{I}$ as $\mathbb{I}_n \times \mathbb{M}_0(\kappa_{n-1}, \kappa_n) \times \mathbb{I}^{n+1}$. The poset $\mathrm{Add}(\omega, \sup_n \kappa_n) \times \mathbb{T}^{n+1}$ (where $\mathbb{T}^{n+1}$ is ordered by $s(R, 1_{\mathbb{P}})$) projects onto $\mathbb{I}^{n+1}$. Thus by Lemma 2.1.14 the poset

$$\mathbb{I}^p := \mathbb{I}_n \times \mathbb{M}_0(\kappa_{n-1}, \kappa_n) \times \mathrm{Add}(\omega, \sup_n \kappa_n) \times \mathbb{T}^{n+1}$$

projects onto $\mathbb{I}$. Let $\mathbb{Q}$ denote the quotient of $\mathbb{I}^p$ by $\mathbb{I}$.

Claim. *After forcing with $\mathbb{I}$, $\mathbb{Q}$ is $<\kappa_n = \omega_{n+2}$-distributive.*

Proof. By Lemma 2.1.14, $\mathbb{Q}$ is equal to the quotient of $\mathrm{Add}(\omega, \sup_n \kappa_n) \times \mathbb{T}^{n+1}$ by $\mathbb{I}^{n+1}$. By Lemma 4.1.11, this quotient is $<\kappa_n$-distributive since the poset $\mathbb{I}^{n+1}$ has a κ_n-strategically closed term ordering and a c.c.c. base ordering. □

So it suffices to show that $\mathbb{I}^p$ forces $\mathrm{GDSC}(\omega_{n+2})$. Forcing with $\mathbb{I}^p$ can be regarded as forcing first with $\mathbb{T}^{n+1}$, then with $\mathbb{M}_0(\kappa_{n-1}, \kappa_n)$ (this forcing has the same definition after forcing with $\mathbb{T}^{n+1}$ due to the closure of $\mathbb{T}^{n+1}$) and then with $\mathbb{I}_n \times \mathrm{Add}(\omega, \sup_n \kappa_n)$. After forcing with $\mathbb{T}^{n+1}$, κ_n is still Mahlo (again due to the closure) and so forcing with $\mathbb{M}_0^V(\kappa_{n-1}, \kappa_n) = \mathbb{M}_0^{V[\mathbb{T}^{n+1}]}(\kappa_{n-1}, \kappa_n)$ in that model forces $\mathrm{GDSC}(\kappa_{n-1}^+)$. After forcing with $\mathbb{T}^{n+1} \times \mathbb{M}_0(\kappa_{n-1}, \kappa_n)$, $\mathbb{I}_n \times \mathrm{Add}(\omega, \sup_n \kappa_n)$ is κ_{n-1}-c.c., so it preserves $\mathrm{GDSC}(\kappa_{n-1}^+)$. As it forces $\kappa_{n-1} = \omega_{n+1}$, in summary, $\mathbb{I}^p$ forces $\mathrm{GDSC}(\omega_{n+2})$. □

A small variation shows the following: Note that we cannot simply apply Theorem 7.1.2 as $2^\omega = (\sup_n \kappa_n)^+$ after forcing with $\mathbb{I}$.

Theorem 7.2.16. $\mathbb{I}$ *forces that there exists a disjoint stationary sequence on every ω_{n+2}.*

Proof. As shown in Theorem 7.2.4, forcing with $\mathbb{M}_0(\kappa_{n-1}, \kappa_n)$ adds a disjoint stationary sequence $(\mathcal{S}_\alpha)_{\alpha \in S}$ on κ_n, i.e. $S \subseteq \kappa_n \cap \mathrm{cof}(\kappa_{n-1})$ is stationary. Forcing with the κ_{n-1}-c.c. poset $\mathbb{I}_{n-1} \times \mathrm{Add}(\omega, \sup_n \kappa_n)$ preserves that $(\mathcal{S}_\alpha)_{\alpha \in S}$ is a disjoint stationary sequence and moreover forces $\kappa_n = \omega_{n+2}$. After forcing with the κ_n-c.c. poset $\mathbb{I}_n \times \mathrm{Add}(\omega, \sup_n \kappa_n)$, $\mathbb{T}^{n+1}$ is still strongly $<\kappa_n$-distributive, so it preserves the stationarity of S as well as of every $\mathcal{S}_\alpha$. Ergo $(\mathcal{S}_\alpha)_{\alpha \in S}$ is a disjoint stationary sequence in $\mathbb{I}_n \times \mathrm{Add}(\omega, \sup_n \kappa_n) \times \mathbb{T}^{n+1}$. As the latter poset projects onto $\mathbb{I}$ and being a disjoint stationary sequence is downwards absolute, we are done. □

7.3 Distinguishing Internal Clubness and Approachability

In this section, we will introduce a variant of Mitchell forcing which, by virtue of collapsing cardinals in a particular way, adds stationarily many sets which are internally club but not internally approachable. This is joint work with Maxwell Levine. The idea of using this particular collapsing poset is due to Krueger (from [Kru07]).

7.3.1 A Single Cardinal

We intend to distinguish internal clubness and approachability as follows: At stage δ, we aim to collapse δ in such a way that $H(\delta)[G']$ (where G' is generic for the forcing until δ) is internally club. To ensure that the set is not internally approachable, we need our collapsing sequence to not be "fresh" over $V[G']$ (i.e. it cannot have all of its initial segments in that model). This means that we first have to force with some small poset (in our case, $\mathrm{Add}(\tau)$). However (at least in the case of $\tau = \omega$), this poset forces the new sets to be stationary, thus violating the requirement that $H(\delta)[G']$ needs to be internally club. This is fixed by forcing the old sets to be club again by shooting a club through $[H(\delta)[G']]^{<\mu} \cap V[G']$. That is why we need the specific collapse defined in Definition 2.2.5.

Definition 7.3.1. Let $\tau < \mu < \kappa$ be regular cardinals such that $\tau^{<\tau} = \tau$ and κ is inaccessible. Let $A := \kappa$ and define F by induction on γ. If $\gamma = \delta+1$ for an inaccessible cardinal δ, let $F(\gamma)$ be an $\mathbb{M}(\tau, \mu, \kappa, A, F, \gamma)$-name for $\mathbb{P}([\delta]^{<\mu} \cap V[\mathbb{M}(\tau, \mu, \kappa, A, F, \delta)])$. Otherwise, let $F(\gamma) := \check{1}$.

Define $\mathbb{M}_1(\tau, \mu, \gamma) := \mathbb{M}(\tau, \mu, \kappa, A, F, \gamma)$. If $\nu < \kappa$ and G is $\mathbb{M}(\tau, \mu, \kappa, A, F, \nu)$-generic, let $\mathbb{M}_1(G, \tau, \mu, \kappa \setminus \nu) := \mathbb{M}(G, \tau, \mu, \kappa \setminus \nu, A, F)$. Define $\mathbb{T}_1(\tau, \mu, \kappa)$ and $\mathbb{T}_1(G, \tau, \mu, \kappa \setminus \nu)$ accordingly.

For the rest of this section, we fix regular cardinals $\tau < \mu < \kappa$ such that $\tau^{<\tau} = \tau$ and κ is inaccessible.

The following lemma was first noticed by Krueger (albeit with a different proof). The main point is that, even though $\mathbb{P}(S)$ is in many cases not μ-strategically closed since it destroys the stationarity of a subset of $[\delta]^{<\mu}$, the term ordering on $\mathrm{Add}(\tau) * \mathbb{P}([\delta]^{<\mu} \cap V)$ is well-behaved:

Lemma 7.3.2. *Let $\delta > \mu$ be regular. The term ordering on $\mathrm{Add}(\tau) * \mathbb{P}([\delta]^{<\mu} \cap V)$ is μ-strategically closed.*

Proof. It suffices to find a winning strategy for COM in $(\mathbb{P}([\delta]^{<\mu} \cap V), s(R, p))$ for every $p \in \mathrm{Add}(\tau)$, so let $p \in \mathrm{Add}(\tau)$ be arbitrary. At any stage γ, COM will play $\dot{q}_\gamma$ such that the following holds:

1. There is ν_γ such that $p \Vdash \mathrm{dom}(\dot{q}_\gamma) = \check{\nu}_\gamma + 1$
2. There is x_γ such that $p \Vdash \dot{q}_\gamma(\check{\nu}_\gamma) = \check{x}_\gamma$.

We will do the limit step first as it is easier: If COM has played according to the strategy until γ, we let $\nu_\gamma := \bigcup_{\alpha<\gamma} \nu_\alpha$ and $x_\gamma := \bigcup_{\alpha<\gamma} x_\alpha$. Let $\dot{q}_\gamma$ be such that p forces $\dot{q}_\gamma(\alpha) = \dot{q}_\beta(\alpha)$ for some $\beta < \alpha$ whenever $\alpha < \nu_\gamma$ and $\mathrm{dom}(\dot{q}_\gamma) = \check{\nu}_\gamma + 1$ as well as $\dot{q}_\gamma(\check{\nu}_\gamma) = \check{x}_\gamma$.

Now assume $\gamma = \beta + 1$ is a successor ordinal and INC has just played $\dot{q}_\beta$. Because $\mathrm{Add}(\tau)$ is τ^+-Knaster, $\nu_\gamma := \sup\{\nu \mid \exists p' \leq p(p' \Vdash \mathrm{dom}(\dot{q}_\beta) = \check{\nu})\}$ is below μ and the set $x_\gamma := \{\epsilon \mid \exists p' \leq p(p' \Vdash \check{\epsilon} \in \bigcup \mathrm{im}(\dot{q}_\beta))\}$ has size $< \mu$. Let $\dot{q}_\gamma$ be an $\mathrm{Add}(\tau)$-name for a function with domain $\check{\nu}_\gamma + 1$, defined as follows: $\dot{q}_\gamma$ repeats $\dot{q}_\beta(\mathrm{dom}(\dot{q}_\beta) - 1)$ for every ordinal between $\mathrm{dom}(\dot{q}_\beta) - 1$ and ν_γ. Afterwards, $\dot{q}_\gamma(\check{\nu}_\gamma) := \check{x}_\gamma$. We show that $\dot{q}_\gamma$ is as required: Obviously, $\dot{q}_\gamma$ is forced to extend every $\dot{q}_\beta$. Furthermore, $\dot{q}_\gamma$ is forced to be a function into V. Lastly, there are no "new" limit steps, so $\dot{q}_\gamma$ is forced to be continuous. $\square$

As an easy corollary, we obtain that $\mathbb{P}([\delta]^{<\mu} \cap V)$ is suitable as a collapse for constructing variants of Mitchell forcing.

Corollary 7.3.3. *Let $\nu < \delta < \kappa$ be such that ν, δ are inaccessible. Let G be $\mathbb{M}(\tau, \mu, \kappa, A, F, \nu)$-generic.*

1. *The term ordering on* $\mathbb{M}(\tau, \mu, \kappa, A, F, \delta + 1) * F(\delta + 1)$ *is* μ*-strategically closed.*

2. *The term ordering on* $\mathbb{M}(G, \tau, \mu, \kappa \setminus \nu, A, F, \delta + 1) * F(\delta + 1)$ *is* μ*-strategically closed in* $V[G]$*.*

Proof. We first prove (1). The poset $\mathbb{M}(\tau, \mu, \kappa, A, F, \delta + 1)$ is isomorphic to the two-step iteration $\mathbb{M}(\tau, \mu, \kappa, A, F, \delta) * \mathrm{Add}(\tau)$, so $\mathbb{M}(\tau, \mu, \kappa, A, F, \delta + 1) * F(\delta + 1)$ is isomorphic to $\mathbb{M}(\tau, \mu, \kappa, A, F, \delta) * (\mathrm{Add}(\tau) * \mathbb{P}([\delta]^{<\mu} \cap V))$. By Lemma 7.3.2, $\mathbb{M}(\tau, \mu, \kappa, A, F, \delta)$ forces that the term ordering on $\mathrm{Add}(\tau) * \mathbb{P}([\delta]^{<\mu} \cap V)$ is μ-strategically closed. Let σ be an $\mathbb{M}(\tau, \mu, \kappa, A, F, \delta)$-name for a winning strategy. We now play as follows: Given a sequence $(p, q_\alpha)_{\alpha<\gamma}$ in the term ordering on $\mathbb{M}(\tau, \mu, \kappa, A, F, \delta + 1) * F(\delta + 1)$, let $\sigma'((p, q_\alpha)) = (p, \sigma((p(\delta), q_\alpha)_{\alpha<\gamma}))$. Clearly σ' is a valid strategy.

Now we prove (2). As before, we have that $\mathbb{M}(\tau, \mu, \kappa, A, F, \delta)$ forces that the term ordering on $\mathrm{Add}(\tau) * F(\delta + 1)$ is μ-strategically closed. Using that $\mathbb{M}(\tau, \mu, \kappa, A, F, \delta)$ is equivalent to $\mathbb{M}(\tau, \mu, \kappa, A, F, \nu) * \mathbb{M}(\Gamma, \tau, \mu, \kappa \setminus \nu, A, F, \delta)$ and we are in an extension by $\mathbb{M}(\tau, \mu, \kappa, A, F, \nu)$, $\mathbb{M}(G, \tau, \mu, \kappa \setminus \nu, A, F, \delta)$ forces that the term ordering on $\mathrm{Add}(\tau) * F(\delta+1)^G$ is μ-strategically closed. Now we can proceed as in (1). $\square$

If $\nu < \kappa$ is inaccessible, ν itself is a witness to the existence of some $\nu \in A \setminus \nu$ such that $\mathrm{im}(F \restriction [\nu, \nu]) = \{\check{1}\}$. So by Corollary 6.2.7, we have:

Lemma 7.3.4. *Let* $\nu \in (\mu, \kappa)$ *be inaccessible. Let* G *be* $\mathbb{M}_1(\tau, \mu, \nu)$*-generic. In* $V[G]$*,* $\mathbb{M}_1(G, \tau, \mu, \kappa \setminus \nu)$ *has the* $< \tau^+$*-approximation property.*

Now we are able to show that $\mathbb{M}_1(\tau, \mu, \kappa)$ works as intended.

Theorem 7.3.5. *Let* κ *be a Mahlo cardinal and let* G *be* $\mathbb{M}_1(\tau, \mu, \kappa)$*-generic. In* $V[G]$*, for each* $\Theta \geq \kappa$*, there exist stationarily many* $N \in [H^{V[G]}(\Theta)]^\mu$ *such that* N *is internally club but not internally approachable.*

Proof. In $V[G]$, let $F\colon [H^{V[G]}(\Theta)]^{<\omega} \to [H^{V[G]}(\Theta)]^\mu$. Let $\dot{F}$ be a name for F. As before, our aim is to find $Z \in [H^{V[G]}(\Theta)]^\mu$ which is closed under F and internally stationary but not internally club.

Let Θ' be large enough. Applying the Mahloness of κ in V (see Theorem 5.2.1), find $M \prec H^V(\Theta)$ with the following:

1. $\nu := |M| = M \cap \kappa \in \kappa$ is inaccessible.

2. $\dot{F}, \tau, \mu, \kappa, \Theta \in M$

3. ${}^{<\nu}M \subseteq M$.

Let $\pi\colon M \to N$ be the Mostowski-Collapse of M. By Lemma 5.3.5, the collapse π extends to $\pi\colon M[G] \to N[G']$ in $V[G]$, where $G' := G \cap \mathbb{M}_1(\tau, \mu, \nu)$. Furthermore, in $V[G']$, $N[G']$ is closed under $< \nu$-sequences.

Our aim is to show that $M[G] \cap H^{V[G]}(\Theta)$ is internally club but not internally approachable (it is clearly closed under $F = \dot{F}^G$).

Claim. *$M[G]$ is internally club in $V[G]$.*

Proof. As before, by Lemma 7.1.1, it suffices to show that $N[G']$ is internally club in $V[G]$.

Forcing with $\mathbb{M}_1(\tau, \mu, \nu + 2)$ can be regarded as forcing with $\mathbb{M}_1(\tau, \mu, \nu)$ and then with $\mathrm{Add}(\omega) * \mathbb{P}([\nu]^{<\mu} \cap V[G'])$. Let $H * K$ be the $\mathrm{Add}(\omega) * \mathbb{P}([\nu]^{<\mu} \cap V[G'])$-generic filter induced by G. Then $\bigcup K$ is a continuous, increasing and cofinal function $f \colon \mu \to [\nu]^{<\mu} \cap V[G']$. We know $|N[G']| = |N| = \nu$ via some bijection F (which is in $V[G']$) and so g defined by $g(\alpha) := F[f(\alpha)]$ is a continuous, increasing and cofinal function $g \colon \mu \to [N[G']]^{<\mu} \cap V[G']$. As we stated, $[N[G']]^{<\mu} \cap V[G'] \subseteq N[G']$, so the collection $\{g(\alpha) \mid \alpha \in \mu\}$ is club in $[N[G']]^{<\mu}$ and a subset of $N[G']$. $\square$

This clearly shows that $M[G] \cap H^{V[G]}(\Theta)$ is internally club: Let $c \subseteq M[G]$ be club in $[M[G]]^{<\mu}$. Then $\{m \cap H^{V[G]}(\Theta) \mid m \in c\}$ contains a club in $[M[G] \cap H^{V[G]}(\Theta)]^{<\mu}$ and is contained in $M[G] \cap H^{V[G]}(\Theta)$ (every element is a $<\Theta$-sized subset of $H^{V[G]}(\Theta)$).

Lastly, we show:

Claim. *$M[G] \cap H^{V[G]}(\Theta)$ is not internally approachable in $V[G]$.*

Proof. Assume $(x_i)_{i<\mu}$ is an increasing, continuous and cofinal sequence of elements of $[M[G] \cap H^{V[G]}(\Theta)]^{<\mu}$ such that $(x_i)_{i<j} \in M[G] \cap H^{V[G]}(\Theta)$ for every $j < \mu$. In particular (weakening on both sides and letting $y_i := x_i \cap \kappa$), the sequence $(y_i)_{i<\mu}$ is increasing, continuous and cofinal in $[\nu]^{<\mu}$ (as $\nu = \kappa \cap M[G] \cap H^{V[G]}(\Theta)$, since $\Theta \geq \kappa$) and $(y_i)_{i<j} \in M[G]$ for every $j < \mu$. It follows that $\pi((y_i)_{i<j}) = (y_i)_{i<j}$ and so $(y_i)_{i<j} \in N[G'] \subseteq V[G']$ for every $j < \mu$. As $V[G]$ is an extension of $V[G']$ using a forcing order with the $<\mu$-approximation property, $(y_i)_{i<\mu} \in V[G']$. However, every y_i has size $<\mu$ in $V[G]$ and thus also in $V[G']$ (μ remains a cardinal). So, using $(y_i)_{i<\mu}$, we can define a bijection between μ and ν in $V[G']$, a contradiction, as ν is a cardinal in V and $V[G']$ is an extension of V using a forcing order with the ν-c.c. $\square$

So $M[G] \cap H^{V[G]}(\Theta)$ is a set closed under $\dot{F}^G$ which is internally club but not internally approachable. $\square$

Remark 7.3.6. As we have stated in the beginning, the model $M[G] \cap H^{V[G]}(\Theta)$, which is not internally approachable, is still internally approachable in the weak sense: From the proof of Theorem 5.2.1 we can see that $M = \bigcup_{i<\nu} M_i$, where $(M_i)_{i<\nu}$ is an $\in$-increasing sequence. This implies $M[G] = \bigcup_{i<\nu} M_i[G]$ and $(M_i[G])_{i<\nu}$ is an $\in$-increasing sequence. However, the sequence is both too long and its members are too large to serve as a witness to the (untrue) internal approachability of $M[G]$.

7.3.2 Infinitely Many Cardinals

Similarly to before, we prove two preservation lemmas first.

We first prove a slight strengthening of Theorem 7.3.5. We were unable to show that the distinction between internal clubness and internal approachability for models of size μ

is preserved by μ-c.c. forcing. The strongest result in that direction is that the distinction at μ^+ is preserved by forcing with a $<\mu$-centered forcing, shown by Gitik and Krueger in [GK09]. However, we have a result that is "good enough":

Lemma 7.3.7. *Let $\mathbb{P}$ be μ-Knaster and γ any ordinal. Let $G \times H \times I$ be $\mathbb{M}_1(\tau, \mu, \kappa) \times \mathbb{P} \times \mathrm{Add}(\mu, \gamma)$-generic. In $V[G \times H \times I]$, for each $\Theta \geq \kappa$, there exist stationarily many $N \in [H^{V[G\times H\times I]}(\Theta)]^\mu$ which are internally club but not internally approachable.*

We first show that the approximation property of the tail forcing is preserved by forcing with $\mathbb{P} \times \mathrm{Add}(\mu, \gamma)$.

Lemma 7.3.8. *Let $\mathbb{P}$ be μ-Knaster, γ any ordinal and $\nu \in (\mu, \kappa)$ inaccessible. Let $G \times H \times I$ be $\mathbb{M}_1(\tau, \mu, \nu) \times \mathbb{P} \times \mathrm{Add}(\mu, \gamma)$-generic. In $V[G \times H \times I]$, $\mathbb{M}_1(G, \tau, \mu, \kappa \smallsetminus \nu)$ has the $<\mu$-approximation property.*

Proof. In $V[G]$, $\mathbb{M}_1(G, \tau, \mu, \kappa \smallsetminus \nu)$ is an iteration-like partial order with its base ordering isomorphic to $\mathrm{Add}(\tau, \kappa)^V$ and a μ-strategically closed term ordering. As $\mathrm{Add}(\mu, \gamma)^V$ is $<\mu$-distributive in $V[G]$, the same is true in $V[G \times I]$. In $V[G \times I \times H]$, the term ordering, while no longer μ-strategically closed, is still strongly $<\mu$-distributive because H is generic for a μ-c.c. poset. In $V[G \times I]$, the base ordering of $\mathbb{M}_1(G, \tau, \mu, \kappa \smallsetminus \nu)$ is $\mathrm{Add}(\tau, \kappa \smallsetminus \nu)^V$ which is τ^+-Knaster because $(2^{<\tau}) = \tau$ in $V[G \times I]$. Thus $\mathbb{P} \times \mathrm{Add}(\tau, \kappa \smallsetminus \nu)^V$ is μ-square-c.c. in $V[G \times I]$ and $\mathrm{Add}(\tau, \kappa \smallsetminus \nu)^V$ is μ-square-c.c. in $V[G \times H \times I]$. Now apply Theorem 4.2.2. $\square$

Proof of Lemma 7.3.7. We modify the proof of Theorem 7.3.5.

Write $\mathbb{Q} := \mathbb{M}_1(\tau, \mu, \kappa) \times \mathbb{P} \times \mathrm{Add}(\mu, \gamma)$ and $J := G \times H \times I$. As before, let F be a function from $[H^{V[J]}(\Theta)]^{<\omega}$ to $[H^{V[J]}(\Theta)]^\mu$ and $\dot{F}$ a name for F. In V, let Θ' be large enough and find $M \prec H^V(\Theta')$ with the following properties:

1. $\nu := |M| = M \cap \kappa \in \kappa$ is inaccessible.

2. $\dot{F}, \tau, \mu, \gamma, \kappa, \Theta, \mathbb{Q} \in M$

3. ${}^{<\nu}M \subseteq M$.

Let $\pi \colon M \to N$ be the Mostowski-Collapse of M. By Lemma 5.3.5 and Lemma 5.3.6, π extends to $\pi \colon M[J] \to N[J']$, where $J' := G' \times H' \times I'$ is $\mathbb{M}_1(\tau, \mu, \nu) \times \pi(\mathbb{P}) \times \mathrm{Add}(\mu, \pi(\gamma))$-generic. Furthermore, in $V[J']$, $N[J']$ is closed under $<\nu$-sequences (notice that $\mathbb{M}_1(\tau, \mu, \kappa) \times \mathbb{P} \times \mathrm{Add}(\mu, \gamma)$ is κ-Knaster and $\mathbb{M}_1(\tau, \mu, \nu) \times \pi(\mathbb{P}) \times \mathrm{Add}(\mu, \pi(\gamma))$ is ν-Knaster).

Now we show that $M[J] \cap H^{V[J]}(\Theta)$ is internally club but not internally approachable. To this end, we show:

Claim. *$M[J]$ is internally club in $V[J]$.*

Proof. As before, we work with the Mostowski-Collapse $N[J']$. Forcing with $\mathbb{M}_1(\tau, \mu, \nu + 2)$ adds a continuous, increasing and cofinal function $f \colon \mu \to [\nu]^{<\mu} \cap V[G']$. In $V[J']$, there is a bijection F between ν and $N[J']$ and so g, defined by $g(\alpha) := F[f(\alpha)]$ is a continuous, increasing and cofinal function from μ to $[N[J']]^{<\mu} \cap V[J']$. By the closure of $N[J']$ under $<\mu$-sequences in $V[J']$, we see that $\mathrm{im}(g)$ is contained in $N[J']$, so it witnesses that $N[J']$ is internally club in $V[J]$. $\square$

As before, this implies that $M[J] \cap H^{V[J]}(\Theta)$ is internally club in $V[J]$.

Claim. *$M[J] \cap H^{V[J]}(\Theta)$ is not internally approachable in $V[J]$.*

Proof. As before, assuming the claim fails, there is a sequence $(x_i)_{i<\mu}$ of elements of $[\nu]^{<\mu}$ such that $\bigcup_{i<\mu} x_i = \nu$ and $(x_i)_{i<j} \in N[G' \times H' \times I']$ for every $j < \mu$. In particular, any $(x_i)_{i<j}$ is in $V[G' \times H \times I]$ (notice we dropped two "′"). However, $V[G \times H \times I]$ is an extension of $V[G' \times H \times I]$ using $\mathbb{M}_1(\tau, \mu, \kappa \setminus \nu)$ which has the $<\mu$-approximation property in $V[G' \times H \times I]$, so $(x_i)_{i<\mu} \in V[G' \times H \times I]$. As before, this implies that $|\nu| = \mu$ in $V[G' \times H \times I]$, a contradiction, as $G' \times H \times I$ is generic for a forcing with the ν-c.c. $\square$

So $M[J] \cap H^{V[J]}(\Theta)$ is a set closed under $\dot{F}^J$ which is internally club but not internally approachable. $\square$

Lastly, we argue that we actually obtain a stronger property with better downward preservation, mirroring the argument in the previous section.

Definition 7.3.9. Let $\kappa \leq \Theta$ be cardinals and $N \prec H(\Theta)$ with size μ.

1. We say that N is *κ-internally approachable* if there is a sequence $(x_i)_{i<\mu}$ of elements of $[N \cap \kappa]^{<\mu}$ such that $\bigcup_{i<\mu} x_i = N \cap \kappa$ and $(x_i)_{i<j} \in N$ for every $j < \mu$.
2. We let $\mathrm{GDCA}(\mu^+)$ state that for any $\Theta > \mu$ there are stationarily many $N \in [H(\Theta)]^\mu$ which are internally club but not μ^+-internally approachable.

As before, the proof of Lemma 7.3.5 actually shows the following stronger result:

Lemma 7.3.10. *Let $\mathbb{P}$ be μ-Knaster and γ any ordinal. Let $G \times H \times I$ be a gener filter for $\mathbb{M}_1(\tau, \mu, \kappa) \times \mathbb{P} \times \mathrm{Add}(\mu, \gamma)$. In $V[G \times H \times I]$, $\mathrm{GDCA}(\mu^+)$ holds.*

And this property is more easily preserved downwards:

Lemma 7.3.11. *Assume W is a forcing extension ov V by a forcing order $\mathbb{P}$ which is $<\mu^+$-distributive for some regular μ. Assume that $\mathrm{GDCA}(\mu^+)$ holds in W. Then $\mathrm{GDCA}(\mu^+)$ holds in V.*

Proof. Let G be $\mathbb{P}$-generic with $W = V[G]$. Let $C \subseteq [H^V(\Theta)]^\mu$ be club. As in the proof of Lemma 7.2.13 find $M \in C$ with $M[G] \cap V = M$ such that $M[G] \prec (H^{V[G]}(\Theta), H^V(\Theta), \in)$ is internally club but not μ^+-internally approachable.

Claim. *M is internally club in V.*

Proof. Let $c \subseteq [M[G]]^{<\mu} \cap M[G]$ be club. Then $\{m \cap V \mid m \in c\}$ contains a club d in $[M[G] \cap V]^{<\mu} = [M]^{<\mu}$. By the distributivity of $\mathbb{P}$ we have $d \in V$. Furthermore, we know $m \cap V = m \cap H^V(\Theta) \in M[G] \cap V = M$ for any $m \in c$, so $d \subseteq M$. □

Claim. *M is not μ^+-internally approachable in V.*

Proof. This is again clear. □

So we have produced an element of C which is internally club but not μ^+-internally approachable. □

And we can finish the proof of the distinction at infinitely many cardinals: Fix a sequence $(\kappa_n)_{n\in\omega}$ of Mahlo cardinals. For simplicity let $\kappa_{-2} := \omega$ and $\kappa_{-1} := \omega_1$. We force with the fully supported product

$$\mathbb{P} := \prod_{n\in\omega} \mathbb{M}_1(\kappa_{n-2}, \kappa_{n-1}, \kappa_n)$$

Theorem 7.3.12. *Let G be a $\mathbb{P}$-generic filter. In $V[G]$,* GDCA(ω_{n+2}) *holds for every $n \in \omega$.*

Proof. It is easy to see that after forcing with $\mathbb{P}$, $\kappa_n = \omega_{n+2}$. Now let any n and any $\Theta \geq \kappa_n$ be given. By Lemma 2.1.14, we can project onto $\mathbb{P}$ from the product

$$\mathbb{Q} := \prod_{k\leq n} \mathbb{M}_1(\kappa_{k-2}, \kappa_{k-1}, \kappa_k) \times \mathrm{Add}(\kappa_{n-1}, \kappa_{n+1}) \times \mathbb{T}_1(\kappa_{n-1}, \kappa_n, \kappa_{n+1}) \times \prod_{k>n+1} \mathbb{M}_1(\kappa_{k-2}, \kappa_{k-1}, \kappa_k)$$

Again by Lemma 2.1.14, the quotient of this product by $\mathbb{P}$ is equal to the quotient of $\mathrm{Add}(\kappa_{n-1}, \kappa_{n+1}) \times \mathbb{T}_1(\kappa_{n-1}, \kappa_n, \kappa_{n+1})$ by $\mathbb{M}_1(\kappa_{n-1}, \kappa_n, \kappa_{n+1})$ which is $<\kappa_n$-distributive and κ_{n+1}-c.c. by Lemma 4.1.11. Hence, by Lemma 7.3.11 it suffices to show that $\mathbb{Q}$ forces GDCA(ω_{n+2}).

Forcing with $\mathbb{Q}$ can be regarded as forcing first with the product of $\mathbb{T}_1(\kappa_{n-1}, \kappa_n, \kappa_{n+1})$ and $\prod_{k>n+1} \mathbb{M}_1(\kappa_{k-2}, \kappa_{k-1}, \kappa_k)$ and then with $\prod_{k\leq n} \mathbb{M}_1(\kappa_{k-2}, \kappa_{k-1}, \kappa_k) \times \mathrm{Add}(\kappa_{n-1}, \kappa_{n+1})$. Forcing with the first poset (which is κ_n-strategically closed by Lemma 6.2.1 (2)) preserves both the definition of the second poset and the Mahloness of κ_n. As $\prod_{k<n} \mathbb{M}_1(\kappa_{k-2}, \kappa_{k-1}, \kappa_k)$ is κ_{n-1}-Knaster, Lemma 7.3.7 directly implies that whenever G is $\mathbb{Q}$-generic and $\Theta \geq \kappa$, there are stationarily many $N \in [H^{V[G]}(\Theta')]^{\kappa_{n-1}} = [H^{V[G]}(\Theta')]^{\omega_{n+1}}$ which are internally club but not μ^+-internally approachable. □

7.4 Cascading Variants of Internal Approachability

It is easy to see that if $N \prec H(\Theta)$ is internally unbounded (stationary; club; approachable), the same is true for $N \cap H(\Theta')$ whenever $\Theta' \in N$. In all of our previously constructed examples, the reverse implication is also true due to the following reason: The "approachability type" of some model $M[G]$ depends on how the ground-model size of M was collapsed (i.e. using a fresh vs. non-fresh sequence or by shooting a club into the new or the old sets).

When using models given by Mahlo cardinals, we always have $|M| = M \cap \kappa$, so the way the model is approachable is the same at all levels.

It remains to be seen if this is a constraint of our methods or if there is a mathematical reason. In this section, we will show that the former possibility is the case. By using stronger large cardinal assumptions, we will show that a model can be internally approachable of different types at different levels.

The summary of the idea is as follows: Suppose we are given a model M such that $|M \cap H(\kappa)| = \nu$ and $|M \cap H(\kappa^+)| = \nu^+$ for some inaccessible cardinal ν. Then we can collapse ν by simply using $\mathrm{Coll}(\omega_1, \nu)$ which makes $M \cap H(\kappa)$ internally approachable. Since $|M \cap H(\kappa^+)| = \nu^+$, $|M \cap H(\kappa^+)|$ will not acquire size ω_1 by doing so, ergo we can make another choice regarding the collapse of $M \cap H(\kappa^+)$, thus turning this model e.g. internally club but not internally approachable. For simplicity, we will focus on obtaining models $M \prec H(\kappa^+)$ such that $M \cap H(\kappa)$ is internally approachable and $M \cap H(\kappa^+)$ is internally club but not internally approachable, but the method is very malleable.

We first introduce our large cardinal notion:

Definition 7.4.1. Let $\kappa \leq \lambda$ be cardinals. $\mathrm{Pr}(\kappa, \lambda)$ states that for every $\Theta \geq \lambda$ there are stationarily many $N \in [H(\Theta)]^{<\kappa}$ such that the following holds:

1. $\nu := N \cap \kappa$ is an inaccessible cardinal.
2. $[N \cap \lambda]^{<\nu} \subseteq N$.
3. For every $\mu \in [\kappa^+, \lambda]$, $\mathrm{otp}(N \cap \mu)$ is a cardinal.

In particular, if N witnesses an instance of $\mathrm{Pr}(\kappa, \kappa^+)$ and is sufficiently elementary, $\mathrm{otp}(N \cap \kappa^+) = \nu^+$: It is clearly greater than ν. On the other hand, for any $\alpha \in \kappa^+ \cap N$, N contains a bijection between κ and α by elementarity, which restricts to a bijection between $N \cap \kappa = \nu$ and $N \cap \alpha$, so ν^+ is the only option.

Remark 7.4.2. The exact consistency strength of $\mathrm{Pr}(\kappa, \lambda)$ is unclear. Of course, $\mathrm{Pr}(\kappa, \kappa)$ is equivalent to κ being Mahlo and $\mathrm{Pr}(\kappa, \lambda)$ follows from the λ-ineffability of κ. $\mathrm{Pr}(\kappa, \kappa^+)$ is connected to the so-called *subcompactness* of κ (see e.g. [Zem17]) and has, among other places, been used (under a different name) by Krueger in [Kru05].

We now define our variant of Mitchell forcing:

Definition 7.4.3. Let $\tau < \mu < \kappa$ be cardinals such that $\tau^{<\tau} = \tau$, μ is regular and κ is inaccessible. Let $A := \{\delta + 1 \mid \delta \in \kappa \text{ inaccessible}\}$ and define F by induction:

If $\gamma = \delta$ for an inaccessible cardinal $\delta \in \kappa$, $F(\gamma)$ is an $\mathbb{M}(\tau, \mu, \kappa, A, F, \gamma)$-name for $\dot{\mathrm{C}}\mathrm{oll}(\check{\mu}, \check{\delta})$. If $\gamma = \delta + 2$ for an inaccessible cardinal $\delta \in \kappa$, $F(\gamma)$ is an $\mathbb{M}(\tau, \mu, \kappa, A, F, \gamma)$-name for $\mathbb{P}([\check{\delta^+}]^{<\mu} \cap V[\mathbb{M}(\tau, \mu, \kappa, A, F, \delta + 1)])$. Otherwise, $F(\gamma) := \check{1}$.

Let $\mathbb{M}_2(\tau, \mu, \gamma) := \mathbb{M}(\tau, \mu, \kappa, A, F, \gamma)$. If $\nu < \kappa$ and G is an $\mathbb{M}(\tau, \mu, \kappa, A, F, \nu)$-generic filter, define $\mathbb{M}_2(G, \tau, \mu, \kappa) := \mathbb{M}(G, \tau, \mu, \kappa, A, F)$.

Using all of our previous results, we can show relatively directly that the following theorem holds:

Theorem 7.4.4. *Assume* $\mathrm{Pr}(\kappa, \kappa^+)$ *and* GCH. *After forcing with* $\mathbb{M}_2(\tau, \mu, \kappa)$*,* $\kappa = \mu^+$ *and there are stationarily many* $N \in [H(\mu^{++})]^\mu$ *such that* $N \cap H(\mu^+)$ *is internally approachable and* $N \cap H(\mu^{++})$ *is internally club but not internally approachable.*

Proof. For simplicity, write $\mathbb{M}(\gamma) := \mathbb{M}(\tau, \mu, \gamma)$. Let G be a $\mathbb{M}(\kappa)$-generic filter. For $\gamma < \kappa$, let $G(\gamma)$ be the $\mathbb{M}(\tau, \mu, \gamma)$-generic filter induced by G. In $V[G]$, let F be a function from $[H^{V[G]}(\kappa^+)]^{<\omega}$ to $[H^{V[G]}(\kappa^+)]^\mu$ and let $\dot{F}$ be a name for F. In V, using $\mathrm{Pr}(\kappa, \kappa^+)$ and letting Θ be large enough, find $M \prec H(\Theta)$ such that $\dot{F} \in M$, $\nu := M \cap \kappa$ is inaccessible, $[M \cap \kappa^+]^{<\nu} \subseteq M$ and $\mathrm{otp}(M \cap \kappa^+) = \nu^+$. We will show that $M[G] \cap H^{V[G]}(\kappa^+)$ is as required (it is clearly closed under $\dot{F}^G$). As before, we work with the Mostowski-Collapses. Clearly, the collapse of $M[G] \cap H^{V[G]}(\kappa)$ is equal to $N_\kappa := \pi(M \cap H^V(\kappa))[G(\nu)]$ and the collapse of $M[G] \cap H^{V[G]}(\kappa^+)$ is equal to $N_{\kappa^+} := \pi(M \cap H^V(\kappa))[G(\nu)]$ (where π is the collapsing function of M).

Claim. N_κ *is internally approachable.*

Proof. Since $\mathbb{M}(\nu + 1) \cong \mathbb{M}(\nu) * \dot{\mathrm{Coll}}(\check{\mu}, \check{\nu})$ and

$$|N_\kappa| = |\pi(M \cap H(\kappa))| = |M \cap H(\kappa)| = M \cap \kappa = \nu$$

there exists in $V[G]$ a sequence $(x_i)_{i<\mu}$ of elements of $[N_\kappa]^{<\mu}$ such that $(x_i)_{i<j} \in V[G(\nu)]$ for any $i < \mu$ and $\bigcup_{i<\mu} x_i = N_\kappa$. As before, $[N_\kappa]^{<\mu} \cap V[G(\nu)] \subseteq N_\kappa$, so $(x_i)_{i<j} \in N_\kappa$ for any $j < \mu$ and $(x_i)_{i<\mu}$ witnesses that N_κ is internally approachable. □

Claim. N_{κ^+} *is internally club.*

Proof. We have $\mathbb{M}(\nu + 3) \cong \mathbb{M}(\nu + 1) * \mathrm{Add}(\tau) * \mathbb{P}([\check{\nu^+}]^{<\mu} \cap V[G(\nu + 1)])$. So in $V[G]$ the set $[N_{\kappa^+}]^{<\mu}$ contains a club consisting of elements of $V[G(\nu + 1)]$ (using that $|N_{\kappa^+}| = \nu^+$, so it is not collapsed by $\mathbb{M}(\nu + 1)$). However, any such element is actually in $V[G(\nu)]$ because $\mathbb{M}(\nu + 1)$ does not add any new $<\mu$-sequences by the closure of $\mathrm{Coll}(\mu, \nu)$. As before, $[N_{\kappa^+}]^{<\mu} \cap V[G(\nu)] \subseteq N_{\kappa^+}$ (since $|H^{V[G]}(\kappa^+)| = \kappa^+$ and $[M \cap \kappa^+]^{<\nu} \subseteq M$), so N_{κ^+} is internally club. □

We are done after showing:

Claim. N_{κ^+} *is not internally approachable.*

Proof. Assume toward a contradiction that there is an increasing and continuous sequence $(x_i)_{i<\mu}$ of elements of N_{κ^+} with union N_{κ^+} such that $(x_i)_{i<j} \in N_{\kappa^+}$ for any $j < \mu$. Ergo any $(x_i)_{i<j}$ is in $V[G(\nu)]$ and in particular in $V[G(\nu + 1)]$. However, by Corollary 6.2.7 (note that the role of ν in that lemma is here taken by $\nu + 1$ and we let $\xi := \nu + 1$) the pair $(V[G(\nu + 1)], V[G])$ has the $<\mu$-approximation property so $(x_i)_{i<\mu} \in V[G(\nu + 1)]$. This implies that $|N_{\kappa^+}| = |\nu^+| = \mu$ in $V[G(\nu + 1)]$, a contradiction as $\mathbb{M}(\nu + 1)$ has the ν^+-c.c. (because the GCH holds). □

This finishes the proof. $\square$

We actually have something stronger in $V[\mathbb{M}_2(\tau,\mu,\kappa)]$ (which we will prove soon): By arguments similar to [Cum+18], $\mu^+ \in I[\mu^+]$, so actually almost every $N \in [H(\mu^+)]^\mu$ with $\mathrm{cof}(N \cap \mu^+) = \mu$ is internally approachable. This shows that $V[\mathbb{M}_2(\tau,\mu,\kappa)]$ is a good starting point toward a model where AP_μ holds and we still have a distinction between internal unboundedness and approachability. We will show that this can be straightforwardly obtained by forcing with $\mathrm{Add}(\mu,\kappa^+)^V$ over the previous model. Since we also want to show that the assumption $2^\mu = \mu^+$ is necessary for Theorem 7.1.2, we are here aiming for a distinction between internal stationarity and clubness (and so we use a different collapse and Cohen Forcing on ω).

Definition 7.4.5. Let $\mu < \kappa$ be cardinals such that μ is regular and κ is inaccessible. For $\gamma \leq \kappa$, let $\mathbb{M}_3(\mu,\gamma) := \mathbb{M}(\omega,\mu,\kappa,A,F,\gamma)$, where $A := \{\delta + 1 \mid \delta \in \kappa \text{ is inaccessible}\}$ and F is defined as follows: If $\gamma = \delta$ for a strong limit cardinal $\delta \in \kappa$, $F(\gamma)$ is an $\mathbb{M}(\omega,\mu,\kappa,A,F,\gamma)$-name for $\dot{\mathrm{Coll}}(\check{\mu},\check{\delta})$ and if $\gamma = \delta + 2$ for an inaccessible cardinal $\delta \in \kappa$, $F(\gamma)$ is an $\mathbb{M}(\omega,\mu,\kappa,A,F,\delta)$-name for $\dot{\mathrm{Coll}}(\check{\mu},\check{\delta^+})$. Otherwise, $F(\gamma) := \check{1}$.

Using what we have seen before, it is not hard to see that $\mathbb{M}(\mu,\kappa)$ forces that $\kappa = \mu^+$ and there exist stationarily many $N \in [H(\mu^{++})]^\mu$ such that $N \cap H(\mu^+)$ is internally approachable and N is internally stationary but not internally club. But we are after the following result:

Theorem 7.4.6. *Assume* $\mathrm{Pr}(\kappa,\kappa^+)$ *and* GCH. *After forcing with* $\mathbb{M}_3(\mu,\kappa) \times \mathrm{Add}(\mu,\kappa^+)$*,* $\kappa = \mu^+$ *and the following holds:*

1. $\mu^+ \in I[\mu^+]$ *(so there does not exist a disjoint stationary sequence on* μ^+*).*
2. *There are stationarily many* $N \in [H(\mu^+)]^\mu$ *which are internally stationary but not internally club.*

Proof. For $\gamma \leq \kappa$ and $A \subseteq \kappa^+$, let $\mathbb{M}(\gamma) \times \mathbb{A}(A) := \mathbb{M}(\mu,\gamma) \times \mathrm{Add}(\mu,A)$. Let $G \times I$ be $\mathbb{M}(\kappa) \times \mathbb{A}(\kappa^+)$-generic and, again for $\gamma \leq \kappa, A \subseteq \kappa^+$, let $G(\gamma) \times I(A)$ be the induced filter on $\mathbb{M}(\gamma) \times \mathbb{A}(A)$. Work in $V[G \times I]$.

We first deal with $\mu^+ \in I[\mu^+]$. The "so"-part was shown in Theorem 7.1.5.

By a result of Shelah (see [She91], Lemma 4.4), $\mu^+ \cap \mathrm{cof}(<\mu) \in I[\mu^+]$. Ergo, to show $\mu^+ \in I[\mu^+]$, it suffices to show that $\mu^+ \cap \mathrm{cof}(\mu) \in I[\mu^+]$. In $V[G \times I]$, we have $|[\mu^+]^{<\mu}| = \mu^+$, so we can fix an enumeration $(a_\alpha)_{\alpha<\mu^+}$ of all elements of $[\mu^+]^{<\mu}$. There is a club C such that whenever $\gamma \in C$ has cofinality μ, $(a_\alpha)_{\alpha<\gamma}$ is an enumeration of $[\mu^+]^{<\mu} \cap V[G(\gamma) \times I]$. Let D be the club of all strong limit cardinals in V below κ. We will show that every $\gamma \in D \cap C \cap \mathrm{cof}(\mu)$ is approachable with respect to $(a_\alpha)_{\alpha<\mu^+}$. To this end, let such a γ be given. Then γ has been collapsed by forcing with $\mathrm{Coll}(\mu,\gamma)$ over $V[G(\gamma) \times I]$. Ergo there is a set $A \subseteq \gamma$ with ordertype $\mu = \mathrm{cf}^{V[G \times I]}(\gamma)$ such that $A \cap \beta \in V[G(\gamma) \times I]$ for every $\beta < \gamma$. Hence γ is approachable with respect to $(a_\alpha)_{\alpha<\mu^+}$.

Now we show that there is a distinction between internal stationarity and clubness for elements of $[H(\mu^+)]^\mu$. To this end, let $F\colon [H^{V[G\times I]}(\kappa)]^{<\omega} \to [H^{V[G\times I]}(\kappa)]^\mu$ and let $\dot{F}$ be a name for F. In V, using $\Pr(\kappa, \kappa^+)$ and letting Θ be large, find $M \prec H(\Theta)$ such that $\dot{F} \in M$, M contains all relevant objects, $\nu := M \cap \kappa$ is inaccessible, $[M \cap \kappa^+]^{<\nu} \subseteq M$ and $\operatorname{otp}(M \cap \kappa^+) = \nu^+$. We will show that $N := M[G \times I] \cap H^{V[G \times I]}(\kappa)$ is as required. We note for later that $M[G \times I] \cap V = M$ by the κ-c.c. of $\mathbb{M}_3(\mu, \kappa) \times \operatorname{Add}(\mu, \kappa^+)$.

Claim. *If $f \in [M \cap \kappa^+]^{<\mu} \cap V[G(\nu) \times I(\kappa^+ \cap M)]$, $f \in M[G \times I]$.*

Proof. Let $\dot{f}$ be an $\mathbb{M}(\nu) \times \mathbb{A}(\kappa^+ \cap M)$-name for f. By the ν-c.c. of that forcing, we can code $\dot{f}$ as a $<\nu$-sized subset of $M \cap \kappa^+$ (since $|\mathbb{M}(\kappa) \times \mathbb{A}(\kappa^+)| = \kappa^+$), so $\dot{f} \in M$. Thus $f = \dot{f}^{G \times I} \in M[G \times I]$ (since $\mathbb{M}(\nu) \times \mathbb{A}(\kappa^+ \cap M)$ is a regular suborder of $\mathbb{M}(\kappa) \times \mathbb{A}(\kappa^+)$). $\square$

This enables us to show:

Claim. *N is internally stationary in $V[G \times I]$.*

Proof. By the previous claim and applying Lemma 2.4.6, we see that $[M \cap \kappa^+]^{<\mu} \cap M[G \times I]$ is stationary in $[M \cap \kappa^+]^{<\mu}$ in $V[G \times I]$. By elementarity, since $\mathbb{M}(\kappa) \times \mathbb{A}(\kappa^+)$ forces $\kappa^{<\kappa} = \kappa^+$, $M[G \times I]$ contains a bijection G between κ^+ and $H^{V[G \times I]}(\kappa)$.

Now assume $c \subseteq [N]^{<\mu}$ is club. By elementarity and since $M[G \times I] \cap V = M$, $G \restriction (M \cap \kappa^+)$ maps from $M \cap \kappa^+$ to $M[G \times I] \cap H^{V[G \times I]}(\kappa) = N$. Ergo $\{G^{-1}[m] \mid m \in c\}$ is club in $[M \cap \kappa^+]^{<\mu}$. Thus there is $m \in c$ such that $G^{-1}[m] \in M[G \times I]$. It follows that $m \in M[G \times I]$. Moreover, m is a $<\mu$-sized subset of $H^{V[G \times I]}(\kappa)$, so $m \in M[G \times I] \cap H^{V[G \times I]}(\kappa) = N$. $\square$

To show that N is not internally club, we first show a converse to the first claim:

Claim. *If $f \in [M \cap \kappa^+]^{<\mu} \cap M[G \times I]$, $f \in V[G(\nu) \times I(\kappa^+ \cap M)]$.*

Proof. Assume $\dot{f} \in M$ is an $\mathbb{M}(\kappa) \times \mathbb{I}(\kappa^+)$-name for a $<\mu$-sized subset of $M \cap \kappa^+$. WLOG assume $\dot{f}$ is forced to have a fixed size δ. For any $\alpha < \delta$, the maximal antichain A_α deciding the value of the αth element of $\dot{f}$ is in M. As it has size $<\kappa$, it is a subset of M and thus a subset of $\mathbb{M}(\nu) \times \mathbb{A}(\kappa^+ \cap M)$. Ergo $\dot{f}$ is an $\mathbb{M}(\nu) \times \mathbb{A}(\kappa^+ \cap M)$-name and so $f \in V[G(\nu) \times I(\kappa^+ \cap M)]$. $\square$

Claim. *N is not internally club in $V[G \times I]$.*

Proof. As before, we do this by working with $M \cap \kappa^+$. Assuming the claim fails, let $c \subseteq N$ be club in $[N]^{<\mu}$. By using the bijection G from the second claim, we see that there is $d \subseteq M[G \times I]$ club in $[M \cap \kappa^+]^{<\mu}$.

In particular, using the third claim, $[M \cap \kappa^+]^{<\mu} \cap V[G(\nu) \times I(\kappa^+ \cap M)]$ contains a club in $[M \cap \kappa^+]^{<\mu}$ and, using a bijection between $M \cap \kappa^+$ and ν^+ in V (and moving to a larger forcing extension), $[(\nu^+)^V]^{<\mu} \cap V[G(\nu) \times I]$ contains a club in $[(\nu^+)^V]^{<\mu}$ in $V[G \times I]$. We will show that this is not the case.

Consider the pair $(V[G(\nu+1) \times I], V[G(\nu+2) \times I])$. In $V[G(\nu+2) \times I]$, $\mu^+ = (\nu^+)^V$ since $G(\nu+1) \times I$ is generic for a forcing with the ν^+-c.c. which collapses ν. Moreover, there is of course a real in $V[G(\nu+2) \times I] \setminus V[G(\nu+1) \times I]$ since $\mathbb{M}(\nu+2)$ is isomorphic to $\mathbb{M}(\nu+1) * \operatorname{Add}(\omega)$. Ergo, by Fact 7.2.2 and previous techniques (note that the term ordering on $\mathbb{M}_3(G(\nu+2), \mu, \kappa \setminus \nu+2)$ remains μ-strategically closed after forcing with $\mathrm{Add}(\mu, \kappa^+)^V)$, the set of all $x \in [(\nu^+)^V]^{<\mu}$ which are not in $V[G(\nu+1) \times I]$ is stationary in $[(\nu^+)^V]^{<\mu}$ in $V[G \times I]$, a clear contradiction. $\square$

So N is closed under $\dot{F}^{G \times I}$ and internally stationary but not internally club. $\square$

CHAPTER 8

On the Ineffable Slender Property

In this chapter we will concern ourselves with the principle ISP, a strengthening of the tree property introduced by Christoph Weiß in his PhD thesis (see [Wei10]). Unlike the principle ITP (that stipulates the existence of ineffable branches for *thin* lists), which behaves much like the tree property (and is forced by many forcings which were originally conceived to force TP), it has much stronger implications and is also harder to force in practice: For the tree property, it is enough in most cases to show that a forcing is projected onto from the product of a sufficiently square-c.c. and a sufficiently closed forcing. However, in our case we also need a certain connection between the square-c.c. and the closed component (namely, that the ordering on the product is iteration-like) to ensure that the added sets are not fresh over the intermediate extension. The results in this chapter are due to the author (partially unpublished and from [Jak24b]).

8.1 Two ISP Preservation Theorems

In this section, we will give two criteria for when a forcing order preserves ISP. The latter is a strengthening of the former, but we state and prove both separately to facilitate a better understanding. The proof of the first theorem is adapted from [HLN19] but the result is more general (and better adapted to variants of Mitchell forcing).

Theorem 8.1.1. *Let $\delta \leq \kappa \leq \lambda = \lambda^{<\kappa}$ be regular cardinals and $\mathbb{P}$ a poset. Assume the following:*

1. *$\mathbb{P}$ is of size $\leq \lambda$ and κ-c.c.,*
2. *For every (κ, λ)-list e, every sufficiently large Θ and every $x \in H(\Theta)$, there is a λ-ineffability witness M for κ with respect to e such that $\mathbb{P} \in M$ and the following holds:*
   - *(a) $x \in M$*
   - *(b) $[M \cap \lambda]^{<M\cap\kappa} \subseteq M$.*
   - *(c) Whenever G is $\mathbb{P}$-generic over V, $\pi[G \cap M]$ is $\pi(\mathbb{P})$-generic over V and the pair $(V[\pi[G \cap M]], V[G])$ has the $<\pi(\delta)$-approximation property.*

Then if κ is λ-ineffable, $\mathbb{P}$ forces $\mathrm{ISP}(\check{\delta}, \check{\kappa}, \check{\lambda})$.

Proof. Without loss of generality, consider $\mathbb{P}$ to be a partial order on a subset of λ. Denote by $\langle\cdot,\cdot\rangle$ the Gödel pairing function.

Let $\dot{f}$ be a $\mathbb{P}$-name for a $<\delta$-slender (κ,λ)-list, forced by some p. Let $\dot{F}$ be the function corresponding to the club (in some $[H^{V[\Gamma]}(\Theta')]^{<\kappa}$) witnessing the $<\delta$-slenderness of $\dot{f}$. Let Θ be large so that $H^V(\Theta)$ contains all relevant objects.

We will transform $\dot{f}$ into a ground-model (κ,λ)-list. To this end, let $a\in[\lambda]^{<\kappa}$. We consider two cases:

1. If there exists $M\prec H(\Theta)$ (with Mostowski-Collapse π) such that $M\cap\kappa\in\kappa$, $M\cap\lambda=a$, $[M\cap\lambda]^{<M\cap\kappa}\subseteq M$, $\kappa,\lambda,\dot{f},\dot{F}\in M$ and for some $\pi(\mathbb{P})$-name $\dot{x}_a$ and a condition $p_a\leq p$, $p_a\Vdash\dot{f}(M\cap\lambda)=\pi^{-1}[\dot{x}_a^{\pi[\Gamma\cap M]}]$, let
$$g(a):=\{\langle\alpha,\beta\rangle\mid\alpha,\beta\in\pi[a]\wedge\alpha\Vdash\check{\beta}\in\dot{x}_a\}$$
which is a subset of $\pi[a]$ and
$$e(a):=\pi^{-1}[g(a)]$$
2. Otherwise, let $e(a):=\emptyset$.

Let $M\in[H(\Theta)]^{<\kappa}$ be a λ-ineffability witness for κ with respect to e as in the assumptions of the theorem containing all relevant parameters. Let $\pi\colon M\to N$ be the Mostowski-Collapse and $a:=M\cap\lambda$ as well as $\theta:=\pi[a]$. We will show that case (1) holds. Let G_0 be $\mathbb{P}$-generic containing p and $G_0':=\pi[G_0\cap M]$ (which is $\pi(\mathbb{P})$-generic over V by assumption). By Lemma 5.3.5, π extends to $\pi\colon M[G_0]\to N[G_0']$ and $[\theta]^{<\nu}\cap V[G_0']\subseteq N[G_0']$. Furthermore, $M[G_0]\prec H^{V[G_0]}(\Theta)$, so $M[G_0]\cap H^{V[G_0]}(\Theta')$ is closed under $\dot{F}^{G_0}$ and therefore witnesses the slenderness of $\dot{f}^{G_0}$.

Assume $\pi[\dot{f}^{G_0}(a)]\notin V[G_0']$. By the $<\pi(\delta)$-approximation property there is $z\in V[G_0']$ with ordertype $<\pi(\delta)$ such that $\pi[\dot{f}^{G_0}(a)]\cap z\notin V[G_0']$. We can assume $z\subseteq\pi[a]$, so we have $z\in N[G_0']$ and $\pi^{-1}(z)=\pi^{-1}[z]\in M[G_0]$. Because $M[G_0]\cap H^{V[G_0]}(\Theta')$ witnesses the slenderness of $\dot{f}^{G_0}$, we have
$$\pi[\dot{f}^{G_0}(a)]\cap z=\pi[\dot{f}^{G_0}(a)]\cap\pi[\pi^{-1}[z]]=\pi[\underbrace{\dot{f}^{G_0}(a)\cap\pi^{-1}[z]}_{\in M[G_0],\ \subseteq M[G_0]}]=\pi(\dot{f}^{G_0}(a)\cap\pi^{-1}[z])\in N[G_0']$$
a contradiction, as $N[G_0']\subseteq V[G_0']$. So $\pi[\dot{f}^{G_0}(a)]\in V[G_0']$. Ergo there is $p_a\leq p$ as well as a $\pi(\mathbb{P})$-name $\dot{x}_a$ such that $p_a\Vdash\pi[\dot{f}(a)]=\dot{x}_a^{\pi[\Gamma\cap M]}$ which is what we wanted to show.

Now our aim is to show that p_a forces $M[\Gamma]$ to be a λ-ineffability witness for κ with respect to $\dot{f}$. So let G_1 be a $\mathbb{P}$-generic filter containing p_a. It is clear that $\kappa,\lambda,\dot{f}^{G_1}\in M[G_1]$. By assumption there is $b_e\in M$ such that $b_e\cap M=e(M\cap\lambda)$. Define
$$b_f:=\{\beta\in\lambda\mid\exists\alpha\in G_1\ \langle\alpha,\beta\rangle\in b_e\}\in M[G_1]$$
all that is left is to show $b_f\cap M=\dot{f}^{G_1}(M\cap\lambda)=\dot{f}^{G_1}(M[G_1]\cap\lambda)$ (where the last equality holds by Lemma 5.3.5).

Let $\beta \in b_f \cap M$. By elementarity there is $\alpha \in G_1 \cap M$ such that $\langle \alpha, \beta \rangle \in b_e$ and we have $\langle \alpha, \beta \rangle \in b_e \cap M = e(a) = \pi^{-1}[g(a)]$. So $\pi(\langle \alpha, \beta \rangle) = \langle \pi(\alpha), \pi(\beta) \rangle \in g(a)$. By the definition, $\pi(\alpha) \Vdash \pi(\check{\beta}) \in \dot{x}_a$. Hence $\pi(\beta) \in \dot{x}_a^{\pi[G_1 \cap M]}$ and $\beta \in \dot{f}^{G_1}(a)$.

Let $\beta \in \dot{f}^{G_1}(M \cap \lambda)$, so $\pi(\beta) \in \dot{x}_a^{\pi[G_1 \cap M]}$. Thus there exists $\pi(\alpha) \in \pi[G_1 \cap M]$ such that $\pi(\alpha) \Vdash \pi(\beta) \in \dot{x}_a$. Hence $\pi(\langle \alpha, \beta \rangle) \in g(a)$ and $\langle \alpha, \beta \rangle \in e(a) = b_e \cap M$. Ergo $\beta \in b_f \cap M$, since $\alpha \in G_1$. $\square$

Now we state the second result which can be seen as an improvement of the first one by allowing more complicated "lifting" arguments:

Theorem 8.1.2. *Let $\delta \leq \kappa \leq \lambda = \lambda^{<\kappa}$ be regular cardinals. Let $\mathbb{P}$ be a poset and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a poset. Define $\mathbb{R} := \mathbb{P} * \dot{\mathbb{Q}}$. Assume the following:*

1. *$\mathbb{R}$ is of size $\leq \lambda$ and $\mathbb{P}$ is κ-c.c.*
2. *$\mathbb{P}$ forces $\dot{\mathbb{Q}}$ to be $<\kappa$-distributive.*
3. *For every (κ, λ)-list e, every sufficiently large Θ, every $x \in H(\Theta)$ and every $r \in \mathbb{R}$ there is a λ-ineffability witness M for κ with respect to e with $\mathbb{P} * \dot{\mathbb{Q}} \in M$ and a condition $r' \leq r$ such that the following holds:*
   (a) *$x \in M$*
   (b) *$[M \cap \lambda]^{<M \cap \kappa} \subseteq M$.*
   (c) *Whenever K is $\mathbb{R}$-generic over V containing r', $\pi[K \cap M]$ is $\pi(\mathbb{R})$-generic over V and the pair $(V[\pi[K \cap M]], V[K])$ has the $<\pi(\delta)$-approximation property.*

Then if κ is λ-ineffable, $\mathbb{R}$ forces $\mathrm{ISP}(\check{\delta}, \check{\kappa}, \check{\lambda})$.

Proof. Let $\dot{f}$ be forced by some $r \in \mathbb{R}$ to be a $<\delta$-slender (κ, λ)-list (witnessed by some $\dot{F}$). Let Θ be large. We call $M \prec H(\Theta)$ *suitable* if the following holds (letting $\pi\colon M \to N$ be the Mostowski-Collapse):

1. $\nu_M := M \cap \kappa \in \kappa$ is inaccessible,
2. $p, \kappa, \lambda, \dot{f}, \dot{F}, \mathbb{R} \in M$,
3. $[M \cap \lambda]^{<M \cap \kappa} \subseteq M$,
4. there is a condition $r_M \leq r$ such that whenever K is $\mathbb{R}$-generic over V containing p_M, $\pi[K \cap M]$ is $\pi(\mathbb{R})$-generic over V and the pair $(V[\pi[K \cap M]], V[K])$ has the $<\pi(\delta)$-approximation property.

Now we define the following (κ, λ)-list e: Let $a \in [\lambda]^{<\kappa}$.

1. If there exists a suitable $M \prec H(\Theta)$ (with Mostowski-Collapse π) with $a = M \cap \lambda$ and a condition $r_a \leq r_M$ as well as a $\pi(\mathbb{R})$-name $\dot{x}_a$ such that $r_a \Vdash \dot{f}(M \cap \lambda) = \pi^{-1}[\dot{x}_a^{\pi[\Gamma \cap M]}]$, let
$$g(a) := \{\langle \alpha, \beta \rangle \mid \alpha, \beta \in \pi[a] \wedge \alpha \Vdash \check{\beta} \in \dot{x}_a\}$$
and
$$e(a) := \pi^{-1}[g(a)]$$
2. Otherwise, let $e(a) := \emptyset$.

Let $M \prec H(\Theta)$ be a λ-ineffability witness for κ with respect to e as in the requirements. Clearly M is suitable. We will show that case (1) holds for $a := M \cap \lambda$. Define $\theta := \pi[a]$. Let $G_0 * H_0$ be $\mathbb{R} = \mathbb{P} * \dot{\mathbb{Q}}$-generic containing r_M. By assumption, $G_0' * H_0' := \pi[G_0 * H_0 \cap M]$ is $\pi(\mathbb{R}) = \pi(\mathbb{P}) * \pi(\dot{\mathbb{Q}})$-generic over V and so by Lemma 5.3.4 and Lemma 5.3.6, π extends to $\pi \colon M[G_0 * H_0] \to N[G_0' * H_0']$. By Lemma 5.3.5, $[\delta]^{<\nu} \cap V[G_0'] \subseteq N[G_0']$ and because H_0' is generic for a $<\nu$-distributive partial order, $[\delta]^{<\nu} \cap V[G_0' * H_0'] \subseteq N[G_0' * H_0']$.

Now we can proceed just as in the proof of Theorem 8.1.1 to show (1) that r_a exists and (2) that r_a forces $M[\Gamma]$ to be a λ-ineffability witness for κ with respect to $\dot{f}$. $\square$

8.2 ISP at Weakly Inaccessible Cardinals

In [Moh23], Mohammadpour asked whether $\mathrm{ISP}(\delta, \kappa, \lambda)$ could hold for κ weakly (but not strongly) inaccessible. In this small section, we answer this question.

Theorem 8.2.1. *Let $\tau < \kappa \leq \lambda \leq \lambda_0 = \lambda_0^{<\kappa}$ be regular cardinals such that κ is λ_0-ineffable and $\tau^{<\tau} = \tau$. $\mathrm{Add}(\tau, \lambda)$ forces that $\mathrm{ISP}(\tau^+, \kappa, \lambda_0)$ holds and κ is weakly, but not strongly, inaccessible.*

Proof. After forcing with $\mathrm{Add}(\tau, \lambda)$, clearly $\tau < \kappa$ and $2^\tau \geq \kappa$, so κ is not strongly inaccessible. Because $\mathrm{Add}(\tau, \lambda)$ does not collapse any cardinals, κ is still weakly inaccessible. Lastly, whenever $M \prec H(\Theta)$ is of size $<\kappa$ with $M \cap \kappa \in \kappa$, $\lambda \in M$, $\tau \subseteq M$ and G is $\mathrm{Add}(\tau, \lambda)$-generic, $G' := \pi[G \cap M]$ is $\mathrm{Add}(\tau, \pi(\lambda))$-generic. We obtain $V[G]$ from $V[G']$ by forcing with $\mathrm{Add}(\tau, \lambda \setminus \pi(\lambda))$ (and moving coordinates around). $\mathrm{Add}(\tau, \lambda \setminus \pi(\lambda))$ is τ^+-Knaster in $V[G']$ and ergo has the $<\tau^+$-approximation property. By Theorem 8.1.1 and Lemma 5.2.4, $\mathrm{Add}(\tau, \lambda)$ forces $\mathrm{ISP}(\tau^+, \kappa, \lambda_0)$. $\square$

8.3 Specifying the Slenderness below κ

In this section we will define a forcing $\mathbb{M}_4(\tau, \mu, \kappa)$ which makes κ into the successor of μ and forces $\mathrm{ISP}(\tau^+, \kappa, \lambda)$ (if κ is λ-ineffable). Additionally, it will force $\neg\,\mathrm{ISP}(\tau, \kappa, \kappa)$. Thus we are able to show that ISP is strictly increasing in strength in the first coordinate. The forcing is almost the same as the one we used in Section 7.2 with the only difference being that we let τ vary.

For this section, fix regular cardinals $\tau < \mu < \kappa$ such that $\tau^{<\tau} = \tau$.

Definition 8.3.1. Let $A := \kappa$ and define F by induction. For any γ of the form $\delta + 1$, where δ is a cardinal, let $F(\gamma)$ be an $\mathbb{M}(\tau, \mu, \kappa, A, F, \gamma)$-name for $\dot{\mathrm{Coll}}(\check{\mu}, \check{\delta})$. Otherwise, let $F(\gamma) := \check{1}$.

Define $\mathbb{M}_4(\tau, \mu, \gamma) := \mathbb{M}(\tau, \mu, \kappa, A, F, \gamma)$. If $\nu \leq \kappa$ and G is $\mathbb{M}_4(\tau, \mu, \nu)$-generic, define $\mathbb{M}_4(G, \tau, \mu, \kappa \smallsetminus \nu) := \mathbb{M}(G, \tau, \mu, \kappa \smallsetminus \nu, A, F)$.

We fix objects as in the above definition. If $\nu < \kappa$ and $\gamma \in [\nu, \kappa) \cap \mathrm{supp}(F)$, $\mathbb{M}(\tau, \mu, \kappa, A, F, \gamma)$ forces $F(\gamma)$ to be $<\mu$-closed. Hence $\mathbb{M}(\tau, \mu, \kappa, \nu) * \mathbb{M}(\Gamma, \tau, \mu, \kappa \smallsetminus \nu, \gamma)$ forces the same and, in $V[G]$, where G is $\mathbb{M}(\tau, \mu, \kappa, \nu)$-generic, $\mathbb{M}(\Gamma, \tau, \mu, \kappa \smallsetminus \nu, \gamma)$ forces $F(\gamma)^G$ to be $<\mu$-closed, ergo the term ordering is $<\mu$-closed. Lastly, if ν is a cardinal, $\nu \in A \smallsetminus \nu$ and $\mathrm{im}(F \restriction [\nu, \nu]) = \{\check{1}\}$. Thus:

Corollary 8.3.2. *If $\nu < \kappa$ is a cardinal and G is $\mathbb{M}_4(\tau, \mu, \nu)$-generic, $\mathbb{M}_4(G, \tau, \mu, \kappa \smallsetminus \nu)$ has the $<\tau^+$-approximation property.*

The last thing left to show is

Theorem 8.3.3. *Let $\lambda = \lambda^{<\kappa} \geq \kappa$ be a regular cardinal such that κ is λ-ineffable. Then $\mathbb{M}_4(\tau, \mu, \kappa)$ forces $\mathrm{ISP}(\tau^+, \kappa, \lambda)$.*

Proof. We show that $\mathbb{M}_4(\tau, \mu, \kappa)$ satisfies the requirements of Theorem 8.1.1. To this end, let e be a (κ, λ)-list and Θ large. Let M be a λ-ineffability witness for κ with respect to e containing τ, μ as in Lemma 5.2.4, i.e. M satisfies the following:

1. M contains all relevant parameters.
2. $\nu := M \cap \kappa \in \kappa$ is inaccessible.
3. $[M \cap \lambda]^{<M \cap \kappa} \subseteq M$.

Then $G' := \pi[G \cap M] = G \cap M = G \cap \mathbb{M}_4(\tau, \mu, \nu)$ is $\mathbb{M}_4(\tau, \mu, \nu)$-generic over V. $V[G]$ is an extension of $V[G']$ by $\mathbb{M}_4(G', \tau, \mu, \kappa \smallsetminus \nu)$ which has the $<\pi(\tau^+) = \tau^+$-approximation property by Lemma 8.3.2. □

Now we show that $\mathbb{M}_4(\tau, \mu, \kappa)$ controls ISP exactly.

Lemma 8.3.4. *$\mathbb{M}_4(\tau, \mu, \kappa)$ forces that $\mathrm{ISP}(\tau, \kappa, \kappa)$ fails.*

We prove a more general statement:

Lemma 8.3.5. *Assume $\delta < \kappa$ are cardinals such that $2^\delta \geq \kappa$ and $2^{<\delta} < \kappa$. Then $\mathrm{ISP}(\delta, \kappa, \kappa)$ fails.*

Proof. Let $(x_\alpha)_{\alpha<\kappa}$ enumerate different subsets of δ. Let $e\colon [\kappa]^{<\kappa} \to [\kappa]^{<\kappa}$ be defined by $e(a) := x_a$ if a is an ordinal above δ and $e(a) := \emptyset$ otherwise.

Claim. *e is $<\delta$-slender.*

Proof. Let Θ be large and C be the club of all $M \in [H(\Theta)]^{<\kappa}$ such that $[\delta]^{<\delta} \subseteq M$ (here we use $2^{<\delta} < \kappa$). If $M \in C$ and $x \in [\kappa]^{<\delta}$, $e(M \cap \kappa) \cap x \in [\delta]^{<\delta}$ (since $e(M \cap \kappa)$ is a subset of δ) and thus in M. $\square$

Claim. *e does not have an ineffable branch.*

Proof. Otherwise there is b such that $S := \{a \in [\kappa]^{<\kappa} \mid e(a) = b \cap a\}$ is stationary. In particular $S \cap \kappa$ is stationary in κ. However, since $e(a)$ is a subset of δ for every a, e is constant on $(S \cap \kappa) \setminus \delta$ which is an obvious contradiction. $\square$

Thus we have produced a $<\delta$-slender (κ, κ)-list without an ineffable branch. $\square$

Lemma 8.3.4 follows because $\tau^{<\tau} = \tau < \kappa$ in V, which is still true in $V[G]$ by the $<\tau$-closure of $\mathbb{M}_4(\tau, \mu, \kappa)$ and $2^\tau = \kappa$ in $V[G]$. We obtain another Corollary:

Corollary 8.3.6. *Assume κ is regular, not a strong limit and $\mathrm{ISP}(\delta, \kappa, \kappa)$ holds. Then $2^{<\delta} = \kappa$.*

Proof. Let $\mu < \kappa$ be minimal such that $2^\mu \geq \kappa$. If $\mu < \delta$ we are done so assume $\mu \geq \delta$. Then $2^{<\mu} < \kappa$ and $2^\mu \geq \kappa$, hence $\mathrm{ISP}(\mu, \kappa, \kappa)$ fails. But since $\mu \geq \delta$, $\mathrm{ISP}(\delta, \kappa, \kappa)$ fails as well since every $<\mu$-slender list is also $<\delta$-slender. $\square$

8.4 ISP and Cardinal Arithmetic

In this section we give easier constructions of two results which were known at ω_2: Lambie-Hanson and Stejskalová showed in [LHS24b] that $\mathrm{ISP}(\omega_1, \omega_2, \geq \omega_2)$ implies $2^\omega = 2^{\omega_1}$ provided $\mathrm{cf}(2^\omega) \neq \omega_1$. Methods in that paper (showing the consistency of $\mathrm{ISP}(\omega_2, \omega_2, \geq \omega_2)$ together with the existence of a Kurepa tree) also show that $\mathrm{ISP}(\omega_2, \omega_2, \geq \omega_2)$ is consistent together with $2^\omega = \omega_2 < 2^{\omega_1}$. In [CK16], Cox and Krueger show that $\mathrm{ISP}(\omega_1, \omega_2, \geq \omega_2)$ is consistent together with an arbitrarily large continuum. Using our previously developed methods, we will give easier proofs of the preceding consistency statements and extend them to cardinals above ω_2.

Definition 8.4.1. Let $\tau < \mu < \nu$ be regular cardinals such that $\tau^{<\tau} = \tau$, $\mu^{<\mu} = \mu$ and ν is inaccessible. For any ordinal γ, define $\mathbb{M}_5(\tau, \mu, \nu, \gamma) := \mathbb{M}_4(\tau, \mu, \nu) \times \mathrm{Add}(\mu, \gamma)$.

For the rest of this section, fix regular cardinals $\tau < \mu < \kappa \leq \lambda$ such that $\tau^{<\tau} = \tau$, $\mu^{<\mu} = \mu$ and κ is inaccessible.

For $\mathbb{M}_5$, we can show directly that the “quotient ordering“ has the correct approximation property (of course building on the results for $\mathbb{M}_4$).

Lemma 8.4.2. *Let $\nu \in (\mu, \kappa]$ be an inaccessible cardinal and γ an ordinal. Let $G' \times H'$ be $\mathbb{M}_5(\tau, \mu, \nu, \gamma)$-generic. In $V[G' \times H']$, $\mathbb{M}_4^{V[G']}(G', \tau, \mu, \kappa \setminus \nu) \times \mathrm{Add}^V(\mu, \lambda \setminus \gamma)$ has the $<\nu$-approximation property.*

Proof. $\mathrm{Add}^V(\mu, \lambda \setminus \gamma)$ is ν-Knaster in $V[G' \times H']$ (where $\mu^+ = \nu$) and thus has the $<\nu$-approximation property in that model. Let H'' be $\mathrm{Add}^V(\mu, \lambda \setminus \gamma)$-generic over $V[G' \times H']$. Then $V[G' \times H'][H'']$ is equal to $V[G'][H]$, where H is $\mathrm{Add}^V(\mu, \lambda)$-generic over $V[G']$ (note $\mathrm{Add}(\mu, \lambda)^V$ is $<\mu$-distributive in $V[G']$). In $V[G']$, $\mathbb{M}_4(G', \tau, \mu, \kappa \setminus \nu)$ is iteration-like and has $\mathrm{Add}^V(\tau, \kappa \setminus \nu)$ as its base ordering as well as a μ-strategically closed term ordering. Thus, in $V[G'][H]$, the ordering is still iteration-like (as this property is absolute), the base ordering is τ^+-Knaster and the term ordering is still μ-strategically closed. Hence $\mathbb{M}_4^{V[G']}(G', \tau, \mu, \kappa \setminus \nu)$ has the $<\tau^+$-approximation property in $V[G'][H] = V[G' \times H'][H'']$ (and thus in particular the $<\nu$-approximation property).

Now let G'' be $\mathbb{M}_4^{V[G']}(G', \tau, \mu, \kappa \setminus \nu)$-generic over $V[G'][H]$ and $G := G' * G''$. Assume there is $f \in V[G \times H]$ such that $f \cap z \in V[G' \times H']$ for every $z \in [V[G' \times H']]^{<\nu} \cap V[G' \times H']$. Let $z \in V[G' \times H]$ have size $<\nu$. Because $\mathrm{Add}^V(\mu, \lambda)$ is ν-Knaster in $V[G' \times H']$ there is $y \in V[G' \times H']$ with $z \subseteq y$ and $|y| < \nu$. Hence $f \cap y \in V[G' \times H']$ and $f \cap z = (f \cap y) \cap z \in V[G' \times H]$. As z was arbitrary and $\mathbb{M}_4^{V[G']}(G', \tau, \mu, \kappa \setminus \nu)$ has the $<\nu$-approximation property in $V[G' \times H]$, $f \in V[G' \times H]$. Now since $\mathrm{Add}^V(\mu, \lambda \setminus \gamma)$ has the $<\nu$-approximation property in $V[G' \times H']$, $f \in V[G' \times H']$. $\square$

A straightforward application of Theorem 8.1.1 shows:

Theorem 8.4.3. *Assume $\lambda_0 \geq \lambda$ is a regular cardinal with $\lambda_0^{<\kappa} = \lambda_0$ such that κ is λ_0-ineffable. Then $\mathbb{M}_5(\tau, \mu, \kappa, \lambda)$ forces $2^\tau = \kappa$, $2^\mu = \lambda$ and that $\mathrm{ISP}(\kappa, \kappa, \lambda_0)$ holds.*

Proof. We again verify the conditions of Theorem 8.1.1. Let e be a (κ, λ_0)-list and M an arbitrary λ_0-ineffability witness for κ with respect to e with $x, \tau, \mu \in M$ as in Lemma 5.2.4. Denote by π the Mostowski-collapse of M and $\nu := \pi(\kappa)$. Then $G' := \pi[G \cap M]$ is $\mathbb{M}_5(\tau, \mu, \nu, \pi(\lambda))$-generic over V. Moreover, $V[G]$ is an extension of $V[G']$ by the forcing $\mathbb{M}_4(G', \tau, \mu, \kappa \setminus \nu) \times \mathrm{Add}(\mu, \lambda \setminus \pi(\lambda))$ (moving coordinates around) which has the $<\nu = \pi(\kappa)$-approximation property by Lemma 8.4.2. So by Theorem 8.1.1 $\mathbb{M}_5(\tau, \mu, \kappa, \lambda)$ forces $\mathrm{ISP}(\kappa, \kappa, \lambda_0)$. $\square$

We also obtain an answer to another question of Mohammadpour which was previously answered in [LHS24b] in the case ω: It is consistent that $\mathrm{ISP}(\kappa, \kappa, \lambda_0)$ holds (with $\kappa = \mu^+$) but $\mathrm{ISP}(\mu, \kappa, \kappa)$ fails:

Lemma 8.4.4. *After forcing with $\mathbb{M}_5(\tau, \mu, \kappa, \lambda)$, $\mathrm{ISP}(\mu, \kappa, \kappa)$ fails.*

Proof. As before, let $f(a)$ be the ath Cohen subset of μ added by $\mathbb{M}_1(\tau, \mu, \kappa, \lambda)$ if a is an ordinal and $\emptyset$ otherwise. Because every $<\mu$-sized segment of $f(a)$ is in $([\mu]^{<\mu})^V$ which has size $\mu < \kappa$, we see that f is μ-slender. However, by previous arguments f cannot have an ineffable branch. $\square$

Now we show that $\mathrm{ISP}(\tau^+, \kappa, \lambda)$ is consistent with an arbitrarily large value for 2^τ.

Definition 8.4.5. Let $\tau < \mu < \nu$ be regular cardinals such that $\tau^{<\tau} = \tau$ and ν is inaccessible. For any ordinal γ, define $\mathbb{M}_6(\tau, \mu, \nu, \gamma) := \mathbb{M}_4(\tau, \mu, \nu) \times \mathrm{Add}(\tau, \gamma)$.

For the rest of this section, we drop the assumption that $\mu^{<\mu} = \mu$. We have a very similar lemma to before (albeit with a stronger approximation property):

Lemma 8.4.6. *Let $\nu \in (\mu, \kappa)$ be inaccessible, γ an ordinal and let $G' \times H'$ be $\mathbb{M}_6(\tau, \mu, \nu, \gamma)$-generic. In $V[G' \times H']$, $\mathbb{M}_4^{V[G']}(G', \tau, \mu, \kappa \setminus \nu) \times \mathrm{Add}^V(\tau, \lambda \setminus \gamma)$ has the $<\tau^+$-approximation property.*

Proof. In $V[G' \times H']$, $\mathrm{Add}^V(\tau, \lambda \setminus \gamma)$ is still $((2^{<\tau})^+)^{V[G' \times H']} = (\tau^+)^{V[G' \times H']}$-square-c.c., so $\mathrm{Add}^V(\tau, \lambda \setminus \gamma)$ has the $<\tau^+$-approximation property in $V[G' \times H']$. Let H'' be $\mathrm{Add}^V(\tau, \lambda \setminus \gamma)$-generic and H the $\mathrm{Add}^V(\tau, \lambda)$-generic filter induced by H' and H''. In $V[G']$, the forcing $\mathbb{M}_4^{V[G']}(G', \tau, \mu, \kappa \setminus \nu)$ has a $<\tau^+$-Knaster base ordering and a μ-strategically closed term ordering. In $V[G' \times H]$, the base ordering is still $<\tau^+$-Knaster and the term ordering is at least $<\mu$-strongly distributive, because $V[G' \times H]$ is an extension of $V[G']$ by a τ^+-Knaster forcing. Being iteration-like is absolute and thus $\mathbb{M}_4^{V[G']}(G', \tau, \mu, \kappa \setminus \nu)$ has the $<\tau^+$-approximation property in $V[G' \times H]$. Now proceed as in Lemma 8.4.2. □

And we can prove:

Theorem 8.4.7. *Let $\tau < \mu < \kappa \leq \lambda \leq \lambda_0 = \lambda_0^{<\kappa}$ be cardinals such that $\tau^{<\tau} = \tau$, μ is regular and κ is λ_0-ineffable. Then $\mathbb{M}_6(\tau, \mu, \kappa, \lambda)$ forces $2^\tau = \lambda$ and that $\mathrm{ISP}(\tau^+, \kappa, \lambda_0)$ holds.*

Proof. This follows just as for $\mathbb{M}_5$. □

8.5 Internally Club Guessing Models

In this subsection, we will introduce a forcing similar to what Krueger used in [Kru09] to obtain a stationary set of structures which are internally club but not internally approachable. Here we will, starting from a λ-ineffable cardinal, construct a model in which for every Θ, if $|H(\Theta)| \leq \lambda$, there are stationarily many $M \in [H(\Theta)]^\mu$ which are $<\tau^+$-guessing models and internally club (in particular, these models are internally club but not internally approachable).

Guessing models were first defined by Viale and Weiß in [VW11]

Definition 8.5.1. Let $\delta \leq \kappa \leq \Theta$ be regular cardinals. $M \in [H(\Theta)]^{<\kappa}$ is a $<\delta$*-guessing model* if $M \prec H(\Theta)$ and whenever x is such that $x \subseteq y$ for $y \in M$ and $x \cap z \in M$ for every $z \in [y]^{<\delta} \cap M$, there is $b \in M$ such that $b \cap M = x \cap M$.

Note that the original definition requires $x \cap z \in M$ for every $z \in [M]^{<\delta} \cap M$, but this is clearly equivalent to the definition given above because $x \cap z = x \cap (y \cap z)$ and $(y \cap z) \in [y]^{<\delta} \cap M$.

Note the similarities between $<\delta$-guessing models and the $<\delta$-approximation property (a forcing $\mathbb{P}$ has the $<\delta$-approximation property if and only if V is $<\delta$-guessing in every extension by $\mathbb{P}$, taking the obvious generalisation).

We introduce the principle related to guessing models:

Definition 8.5.2. Let $\delta \leq \kappa \leq \Theta$ be regular cardinals. The *guessing model principle* $\mathrm{GMP}(\delta, \kappa, \Theta)$ states that the set of $<\delta$-guessing models in $[H(\Theta)]^{<\kappa}$ is stationary.

We have an equivalence between GMP and ISP, originally shown in [VW11]. We will modify the proof slightly to obtain a particular equivalence.

Lemma 8.5.3. *Let $\delta \leq \kappa$ be regular cardinals. The following are equivalent:*

1. $\mathrm{GMP}(\delta, \kappa, \Theta)$ *holds for every regular $\Theta \geq \kappa$.*
2. $\mathrm{ISP}(\delta, \kappa, \lambda)$ *holds for every $\lambda \geq \kappa$.*

Proof. Assume $\mathrm{GMP}(\delta, \kappa, \Theta)$ holds for every $\Theta \geq \kappa$. Let f be a $<\delta$-slender (κ, λ)-list. Let Θ be large such that $f \in H(\Theta)$ and there exists a club $C \subseteq [H(\Theta)]^{<\kappa}$ witnessing $<\delta$-slenderness of f. Let $M \in C$ be a $<\delta$-guessing model containing λ. Because C witnesses slenderness of f, $f(M \cap \lambda) \cap x \in M$ for every $x \in [\lambda]^{<\delta} \cap M$. Because $f(M \cap \lambda) \subseteq \lambda \in M$ and M is $<\delta$-guessing, there is $b \in M$ such that $f(M \cap \lambda) \cap M = f(M \cap \lambda) = b \cap M$. Hence, M is a λ-ineffability witness for κ with respect to f, so f has an ineffable branch.

Assume $\mathrm{ISP}(\delta, \kappa, \lambda)$ holds for every $\lambda \geq \kappa$. Let $\Theta \geq \kappa$ be arbitrary. Let f be the following $(\kappa, H(\Theta))$-list: Let $a \in [H(\Theta)]^{<\kappa}$. Let $f(a) \subseteq a$ to have the property that $f(a) \subseteq g(a)$ for some $g(a) \in a$ and $f(a) \cap z \in a$ for every $z \in [a]^{<\delta} \cap a$, but there is no $b \in a$ such that $f(a) \cap a = b \cap a$ if possible. Otherwise, let $f(a) := \emptyset$.

Claim. *f is $<\delta$-slender.*

Proof. Let Θ' be large and let C consist of all those $M \in [H(\Theta')]^{<\kappa}$ containing $\emptyset$ and $H(\Theta)$ as elements. Let $M \in C$. If $f(M \cap H(\Theta)) = \emptyset$, we are done. Assume otherwise and let $z \in [H(\Theta)]^{<\delta} \cap M$. Then in particular $z \subseteq M$ by its size and so $z \in [H(\Theta) \cap M]^{<\delta} \cap (H(\Theta) \cap M)$. By the definition of f this implies $f(H(\Theta) \cap M) \cap z \in M \cap H(\Theta) \subseteq M$. □

Let $M \prec H(\Theta')$ be an $H(\Theta)$-ineffability witness for κ with respect to f (taking the obvious generalization of Definition 5.2.3).

Claim. *$M \cap H(\Theta)$ is a $<\delta$-guessing model.*

Proof. $M \cap H(\Theta) \prec H(\Theta)$ because $M \prec H(\Theta')$ and $H(\Theta) \in M$ by assumption.

Assume toward a contradiction that $M \cap H(\Theta)$ is not a $<\delta$-guessing model. So there is x with $x \subseteq y$ for some $y \in M \cap H(\Theta)$ and $x \cap z \in M \cap H(\Theta)$ for every $z \in [x]^{<\delta} \cap (M \cap H(\Theta))$

but there is no $b \in M \cap H(\Theta)$ with $b \cap x = (M \cap H(\Theta)) \cap x$. So $x \cap (M \cap H(\Theta))$ is a possible value for $f(M \cap H(\Theta))$ and in particular $f(M \cap H(\Theta)) \neq \emptyset$. Because M is a λ-ineffability witness for κ with respect to f there is $b \in M$ with $b \cap M = f(M \cap H(\Theta))$. Then $b \cap g(M \cap H(\Theta)) \in M \cap H(\Theta)$ and

$$\begin{aligned} f(M \cap H(\Theta)) \cap (M \cap H(\Theta)) &= f(M \cap H(\Theta)) \cap g(M \cap H(\Theta)) \\ &= (b \cap M) \cap g(M \cap H(\Theta)) \\ &= (b \cap g(M \cap H(\Theta))) \cap (M \cap H(\Theta)) \end{aligned}$$

contradicting the choice of $f(M \cap H(\Theta))$. □

So $\mathrm{GMP}(\delta, \kappa, \Theta)$ holds. □

We can extract the following statement from the above proof which will become useful later when producing guessing models with special properties:

Lemma 8.5.4. *Let $\delta \leq \kappa \leq \Theta$ be regular cardinals. There is a $<\delta$-slender $(\kappa, H(\Theta))$-list f such that whenever $M \prec H(\Theta')$ is an $H(\Theta)$-ineffability witness for κ with respect to f, $M \cap H(\Theta)$ is a $<\delta$-guessing model.*

We define our next variant of Mitchell forcing. For this section, fix regular cardinals $\tau < \mu < \kappa$ such that $\tau^{<\tau} = \tau$ and κ is inaccessible. Also fix a function $l \colon \kappa \to \kappa$.

Definition 8.5.5. Let $A := \kappa$ and define $F(l)$ by induction on γ. If $\gamma = \delta + 1$ for an inaccessible cardinal δ and $l(\delta) \geq \delta$, let $F(l)(\gamma)$ be an $\mathbb{M}(\tau, \mu, \kappa, A, F, \gamma)$-name for the poset $\mathbb{P}([l(\delta)]^{<\mu} \cap V[\mathbb{M}(\tau, \mu, \kappa, A, F(l), \delta)])$.

Define $\mathbb{M}_7^l(\tau, \mu, \gamma) := \mathbb{M}(\tau, \mu, \kappa, A, F(l), \gamma)$. If $\nu \leq \kappa$ and G is an $\mathbb{M}(\tau, \mu, \kappa, A, F(l), \nu)$-generic filter, let $\mathbb{M}_7^l(G, \tau, \mu, \kappa \setminus \nu) := \mathbb{M}(G, \tau, \mu, \kappa \setminus \nu, A, F(l))$.

Using a very similar proof to the one for Lemma 7.3.4, we have:

Corollary 8.5.6. *Let $\nu < \kappa$ be inaccessible and G an $\mathbb{M}_7^l(\tau, \mu, \kappa, \nu)$-generic filter. In $V[G]$, $\mathbb{M}_7^l(G, \tau, \mu, \kappa \setminus \nu)$ has the $<\tau^+$-approximation property.*

We can prove the main result of this section:

Theorem 8.5.7. *Let l be a λ-ineffable Laver diamond at κ. Assume* GCH. *If G is $\mathbb{M}_7^l(\tau, \mu, \kappa)$-generic, for every regular $\Theta \in [\kappa, \lambda]$ there are stationarily many $<\tau^+$-guessing models in $H^{V[G]}(\Theta)$ which are internally club.*

Proof. By Lemma 8.5.4 it suffices to show that, in the extension, for every $\Theta \in [\kappa, \lambda]$, every $<\tau^+$-slender $(\kappa, H(\Theta))$-list has an internally club ineffability witness. To this end, let $\dot{f}$ be an $\mathbb{M}_7^l(\tau, \mu, \delta)$-name for a $<\tau^+$-slender $(\kappa, H^{V[\Gamma]}(\Theta))$-list. Let e be the transformed ground-model $(\kappa, H^V(\Theta))$-list (we use that $|H^V(\Theta)| = |H^{V[\Gamma]}(\Theta)|$ which holds as $H^{V[\Gamma]}(\Theta) = H^V(\Theta)[\Gamma]$). Let Θ' be large. Let M be an $H^V(\Theta)$-ineffability witness for κ with respect to

e such that (with Mostowski-Collapse $\pi\colon M \to N$ and $\nu := M \cap \kappa$) $l(\nu) = \pi(\Theta)$ and M is as in Lemma 5.2.4. As in the proof of Theorem 8.1.1, there exists a condition p forcing that $M[\Gamma]$ is an $H^{V[\Gamma]}(\Theta)$-ineffability witness for κ with respect to $\dot{f}$ (using that $H^{V[\Gamma]}(\Theta)$ has the same size as $H^V(\Theta)$). We are done after showing:

Claim. *In $V[G]$, $M[G] \cap H^{V[G]}(\Theta)$ is internally club.*

Proof. By elementarity and since $\Theta^{<\Theta} = \Theta$, because GCH holds in the ground model and the forcing does not increase powerset sizes $\geq \kappa$, $M[G]$ contains a bijection F between Θ and $H^{V[G]}(\Theta)$ (which restricts to a bijection between $M \cap \Theta$ and $M \cap H^{V[G]}(\Theta)$). Forcing with $\mathbb{M}(\tau, \mu, \nu + 2)$ adds an increasing, continuous and cofinal function $f\colon \mu \to [\pi(\Theta)]^{<\mu} \cap V[G']$. By Lemma 5.3.5 we have $[\pi(\Theta)]^{<\mu} \cap V[G'] \subseteq N[G']$, so $\mathrm{im}(f) \subseteq N[G']$. Let g be a function on μ defined by $g(\alpha) := F[\pi^{-1}(f(\alpha))]$ (where π is the extension to $M[G] \to N[G']$). We have $\mathrm{im}(g) \subseteq M[G]$. For any $\alpha < \mu$, $|f(\alpha)| < \mu$, so $f(\alpha) \subseteq N[G']$ and $\pi^{-1}(f(\alpha)) = \pi^{-1}[f(\alpha)]$. Because $\pi \circ F^{-1}\colon M \cap H^{V[G]}(\Theta) \to \pi(\Theta)$ is bijective, g is an increasing, continuous and cofinal function from μ to $[M \cap H^{V[G]}(\Theta)] \cap (M[G] \cap H^{V[G]}(\Theta))$. Ergo g witnesses that $M[G] \cap H^{V[G]}(\Theta)$ is internally club. □

So we have produced an $H^{V[G]}(\Theta)$-ineffability witness for κ with respect to $\dot{f}^G$ which is internally club. □

8.6 Making ISP Indestructible Under Directed-Closed Forcing

It is a well-known result by Laver (see [Lav78]) that if κ is a supercompact cardinal, there is a forcing which leaves κ supercompact and moreover makes the supercompactness of κ indestructible under $<\kappa$-directed closed forcing. Unger showed in [Ung12] that it is possible to force the tree property at ω_2 such that it is indestructible under $<\omega_2$-directed closed forcing. We will adapt his arguments to show that it is consistent from a supercompact cardinal that $\mathrm{ISP}(\tau^+, \kappa, \geq \kappa)$ holds (for κ a successor of a regular cardinal), where $\mathrm{ISP}(\tau^+, \kappa, \geq \kappa)$ means that $\mathrm{ISP}(\tau^+, \kappa, \lambda)$ holds for every $\lambda \geq \kappa$, and is indestructible under $<\kappa$-directed closed forcing.

Our forcing will be a guessing variant of Mitchell forcing, modified to collapse cardinals in a “non-fresh“ way to ensure the approximation property.

We define our next variant. For the rest of this section, fix regular cardinals $\tau < \mu < \kappa$ such that $\tau^{<\tau} = \tau$ and κ is inaccessible. Also fix a function $l\colon \kappa \to V_\kappa$.

Definition 8.6.1. Let A consist of the successors of inaccessible cardinals below κ and define $F(l)$ by induction on γ. If $\gamma \geq \mu$ is an inaccessible cardinal and $l(\gamma)$ is an $\mathbb{M}(\tau, \mu, \kappa, A, F(l), \gamma)$-name for a $<\gamma$-directed closed partial order, let $F(l)(\gamma) := l(\gamma)$. If $\gamma = \delta + 2$ where δ is an inaccessible cardinal, let $F(l)(\gamma)$ be an $\mathbb{M}(\tau, \mu, \kappa, A, F(l), \gamma)$-name for $\mathrm{Coll}(\check{\mu}, \check{\delta})$.

Define $\mathbb{M}_8^l(\tau, \mu, \gamma) := \mathbb{M}(\tau, \mu, \kappa, A, F(l), \gamma)$. If $\xi < \kappa$ and G is an $\mathbb{M}_8^l(\tau, \mu, \kappa, \xi)$-generic filter, let $\mathbb{M}_8^l(G, \tau, \mu, \kappa \setminus \xi) := \mathbb{M}(G, \tau, \mu, \kappa \setminus \xi, A, F(l))$.

Here, we have the approximation property even when forcing with another order after $\mathbb{M}^l_8$. Because the first order used after an inaccessible cardinal ν is $<\gamma$-directed closed and possibly nontrivial (thus it definitely does not have the $<\tau^+$-approximation property), we consider quotients by $\mathbb{M}(\tau,\mu,\xi)$, where ξ is the successor of an inaccessible cardinal.

Corollary 8.6.2. *Let $\xi = \nu + 1$ for an inaccessible cardinal $\nu \geq \mu$. Let G be $\mathbb{M}^l_8(\tau,\mu,\xi)$-generic. In $V[G]$, let $\dot{\mathbb{L}}$ be an $\mathbb{M}^l_8(G,\tau,\mu,\kappa \smallsetminus \xi)$-name for a $<\kappa$-directed closed partial order. Then $\mathbb{M}^l_8(G,\tau,\mu,\kappa \smallsetminus \xi) * \dot{\mathbb{L}}$ has the $<\tau^+$-approximation property.*

Proof. We modify the proof of Corollary 6.2.7. We can regard $\mathbb{M}^l_8(G,\tau,\mu,\kappa\smallsetminus\xi) * \dot{\mathbb{L}}$ as an order on a product by moving $\dot{\mathbb{L}}$ into the second coordinate, i.e. we let $\mathbb{P} := \operatorname{Add}(\tau, A \smallsetminus \xi)$ and $\mathbb{Q}'$ consist of pairs (q,σ) with $\Vdash_{\mathbb{M}^l_8(G,\tau,\mu,\kappa\smallsetminus\xi)} \sigma \in \dot{\mathbb{L}}$, ordered in the natural way.

Claim. *The term ordering on $\mathbb{P}\times\mathbb{Q}$ is μ-strategically closed.*

Proof. Let $\gamma \in \kappa \smallsetminus \xi$. Then in any case, $\mathbb{M}^l_8(G,\tau,\mu,\kappa\smallsetminus\xi,\gamma)$ forces that $F(l)(\gamma)$ is at least μ-strategically closed (possibly trivial or $<\gamma$-directed closed), so the term ordering on the iteration $\mathbb{M}^l_8(G,\tau,\mu,\kappa\smallsetminus\xi,\gamma) * F(l)(\gamma)$ is μ-strategically closed.

Now the claim follows from Lemma 6.2.6. $\square$

Claim. *The term ordering on $\mathbb{P}\times\mathbb{Q}'$ is μ-strategically closed.*

Proof. A winning strategy exists by replying to $(p,(q_\alpha,\sigma_\alpha))_{\alpha<\gamma}$ with $(p,(q_\gamma,\sigma_\gamma))$, where q_γ is played according to the winning strategy for $\mathbb{P}\times\mathbb{Q}$ and σ_γ is forced to be a lower bound of $(\sigma_\alpha)_{\alpha<\gamma}$ (which is forced by (p,q_γ) to be descending). $\square$

Claim. *The ordering on $\mathbb{P}\times\mathbb{Q}'$ is iteration-like.*

Proof. The ordering on $\mathbb{P}\times\mathbb{Q}$ is iteration-like because $\xi+1$ itself is a witness to the existence of some element of A such that $\operatorname{im}(F \restriction [\xi+1,\xi+1]) = \{\check{1}\}$. By standard arguments of names this can be extended to $\mathbb{P}\times\mathbb{Q}'$. $\square$

Now simply apply Theorem 4.2.2.

$\square$

We can show that $\mathbb{M}^l_8(\tau,\mu,\kappa)$ forces ISP in such a way that it is indestructible under $<\kappa$-directed closed forcing.

Theorem 8.6.3. *Let $\tau<\mu<\kappa$ be regular cardinals such that $\tau^{<\tau}=\tau$ and κ is supercompact. Let l be a Laver function. Then $\mathbb{M}^l_8(\tau,\mu,\kappa)$ forces $\mathrm{ISP}(\tau^+,\kappa,\geq\kappa)$ and it is indestructible under $<\kappa$-directed closed forcing.*

Proof. Let $\dot{\mathbb{Q}}$ be an $\mathbb{M}^l_8(\tau,\mu,\kappa)$-name for a $<\kappa$-directed closed forcing and let λ be so large that $\dot{\mathbb{Q}} \in H(\lambda^+)$ (noting that $\mathrm{ISP}(\delta,\kappa,\lambda)$ implies $\mathrm{ISP}(\delta,\kappa,\lambda')$ for $\lambda' \leq \lambda$).

We want to apply Theorem 8.1.2 to show that $\mathbb{M}^l_8(\tau,\mu,\kappa) * \dot{\mathbb{Q}}$ forces $\mathrm{ISP}(\delta,\kappa,\lambda)$. To this end, let e be a (κ,λ)-list, Θ sufficiently large, $x\in H(\Theta)$ and $p \in \mathbb{M}^l_8(\tau,\mu,\kappa) * \dot{\mathbb{Q}}$, $p=(m,\sigma)$.

Because l is in particular a λ-ineffable Laver diamond we can find a λ-ineffability witness M for κ with respect to e such that $\kappa, \dot{\mathbb{Q}}, p, \lambda, \dot{f}, x \in M$, $l(M \cap \kappa) = \pi(\dot{\mathbb{Q}})$ and $[M \cap \lambda]^{<M\cap\kappa} \subseteq M$. Because $|m| < \kappa$, $m \subseteq M$.

Let G be an $\mathbb{M}_8^l(\tau, \mu, \kappa)$-generic filter containing $(m, \pi(\sigma))$ (viewing this as a condition in $\mathbb{M}_8^l(\tau, \mu, \nu+1) \cong \mathbb{M}_8^l(\tau, \mu, \nu) * \pi(\dot{\mathbb{Q}})$). Let $G' := G \cap \mathbb{M}_8^l(\tau, \mu, \nu)$ and let H' be the $\pi(\dot{\mathbb{Q}})^{G'}$-generic filter induced by G (again via the isomorphism). Then $\sigma^G \in \pi^{-1}[H']$ and $\pi^{-1}[H']$ is a $<\kappa$-sized directed subset of $\dot{\mathbb{Q}}^G$ (note $\pi(\dot{\mathbb{Q}}^G) = (\pi(\dot{\mathbb{Q}}))^{G'}$ and $\pi(\sigma^G) = (\pi(\sigma))^{G'}$). Thus there exists r which is below s for every $s \in \pi^{-1}[H']$ (in particular, $r \leq \sigma^G$). If now H is $\mathbb{Q}^G$-generic over $V[G]$ containing r, $\pi[H] = H'$ ($\pi[H]$ is a filter and contains H'; as H' is generic, they are equal). Ergo $\pi[G * H]$ is equivalent to the $\mathbb{M}_8^l(\tau, \mu, \nu+1)$-generic filter induced by G (in particular, it is $\pi(\mathbb{M}_8^l(\tau, \mu, \kappa) * \dot{\mathbb{Q}})$-generic over V) and by Corollary 8.6.2, the pair $(V[\pi[G * H]], V[G * H])$ has the $<\tau^+$-approximation property.

So $(m, \pi(\sigma))$ (as a condition in $\mathbb{M}_8^l(\tau, \mu, \kappa)$) forces the existence of a condition $\dot{r} \in \dot{\mathbb{Q}}$ which forces the required statement. By the maximum principle, $((m, \pi(\sigma)), \dot{r}) \leq (m, \sigma)$ is as required (since $\dot{r}$ is forced to extend σ). Lastly, $\mathbb{M}_8^l(\tau, \mu, \kappa)$ of course forces $\dot{\mathbb{Q}}$ to be $<\kappa$-distributive, so we can apply Theorem 8.1.2. ☐

8.7 ISP and no Disjoint Stationary Sequence

Cox showed in [Cox21] that the conjunction of PFA and $\neg\,\mathrm{DSS}(\omega_2)$ is consistent relative to a supercompact cardinal. Ergo, since PFA implies $\neg\,\mathrm{AP}_{\omega_1}$ it is in particular relatively consistent that AP_{ω_1} and $\mathrm{DSS}(\omega_2)$ fail simultaneously (recall that $\mathrm{DSS}(\omega_2)$ implies $\neg\,\mathrm{AP}_{\omega_1}$). In [Lev24], Levine improved Cox's large cardinal assumptions by showing that $\neg\,\mathrm{AP}_{\omega_1} \wedge \neg\,\mathrm{DSS}(\omega_2)$ is consistent from a Mahlo cardinal (by defining a countable support iteration of proper forcings followed by a forcing to kill any possible disjoint stationary sequence). However, both methods heavily rely on methods which are only available for ω_2. It is therefore natural to wonder whether a similar situation can occur at larger cardinals.

In this section we will prove that when starting from a λ-ineffable cardinal κ and any regular $\tau = \tau^{<\tau} < \mu < \kappa$ there is a forcing extension in which $\kappa = \mu^+ = 2^\tau$, $\mathrm{ISP}(\tau^+, \kappa, \lambda)$ holds (in particular, $\mu^+ \notin I[\mu^+]$) and $\mathrm{DSS}(\mu^+)$ fails. From what we have presented thus far, it should be easy for the reader to modify the proof to show instead that from a Mahlo cardinal κ and any regular $\mu < \kappa$ one can produce a model where $\neg\,\mathrm{AP}_\mu \wedge \neg\,\mathrm{DSS}(\mu^+)$ holds.

We first show the "in particular" statement (which exists in the folklore):

Lemma 8.7.1. *Assume* $\mathrm{ISP}(\mu^+, \mu^+, \mu^+)$ *holds. Then* $\mu^+ \notin I[\mu^+]$.

Proof. To reach a contradiction assume that $\mu^+ \in I[\mu^+]$. This means that there exists a sequence $\overline{x} := (x_\alpha)_{\alpha<\mu^+}$ of elements of $[\mu^+]^{<\mu}$ and a club $C \subseteq \mu^+$ such that for any $\gamma \in C$ there is $A \subseteq \gamma$ unbounded with $\mathrm{otp}(A) = \mathrm{cf}(\gamma)$ such that $\{A \cap \beta \mid \beta < \gamma\} \subseteq \{x_\alpha \mid \alpha < \gamma\}$.

Let f be the following (μ^+, μ^+)-list: If $y \in [\mu^+]^{<\mu^+}$ is an ordinal in C, let $f(y)$ be an $A \subseteq y$ witnessing the approachability of y by $(x_\alpha)_{\alpha<\mu^+}$. Let $f(y) := \emptyset$ otherwise.

f is μ^+-slender: Let Θ be large and let $D \subseteq [H(\Theta)]^{<\mu^+}$ consist of those $M \prec (H(\Theta), \overline{x}, \in)$ such that $M \cap \mu^+ \in C$. Clearly D is club. Let $M \in D$ and $z \in [M \cap \mu^+]^{<M\cap\mu^+} \cap M$. Because $z \in M$, z is bounded in $M \cap \mu^+$ and $\sup(z) =: \beta < M \cap \mu^+$. Ergo $f(M \cap \mu^+) \cap \beta$ is equal to x_α for some $\alpha < \gamma$ since $f(M \cap \mu^+)$ witnesses the approachability of $M \cap \mu^+$ by $(x_\alpha)_{\alpha<\mu^+}$. By elementarity $x_\alpha \in M$ and $f(M \cap \mu^+) \cap z = f(M \cap \mu^+) \cap \beta \cap z \in M$ because M is closed under intersections.

f has no ineffable branch: Assuming $b \subseteq \mu^+$ were an ineffable branch, there would be (by Fodor's lemma) $\gamma < \gamma' \in C$ with equal cofinality such that $f(\gamma) = b \cap \gamma$ and $f(\gamma') = b \cap \gamma'$. However, then $f(\gamma)$ is a proper initial segment of $f(\gamma')$ (because it is unbounded in γ) contradicting the fact that both sets have equal ordertype (namely $\mathrm{cf}(\gamma) = \mathrm{cf}(\gamma')$). $\square$

We will introduce a guessing variant of Mitchell forcing to make $\mathrm{ISP}(\tau^+, \mu^+, \lambda)$ indestructible under many $<\mu$-closed forcings (more specifically, those forcings which allow the lifting of embeddings). Afterwards we will make use of the fact that it is possible to destroy DSS with such a forcing (assuming we prepared the model accordingly), thereby obtaining a model in which $\mathrm{ISP}(\tau^+, \mu^+, \lambda)$ holds but $\mathrm{DSS}(\mu^+)$ fails.

Krueger showed in [Kru09, Proposition 4.1] that the existence of a disjoint stationary sequence is equivalent to the stationarity of a particular set:

Lemma 8.7.2. *Suppose μ is a regular uncountable cardinal and $\mu^{<\mu} \leq \mu^+$. Let $\overline{x} := (x_\alpha)_{\alpha<\mu^+}$ enumerate $[\mu^+]^{<\mu}$ and define*

$$S(\overline{x}) := \{\alpha \in \mu^+ \cap \mathrm{cof}(\mu) \mid [\alpha]^{<\mu} \smallsetminus \{x_\beta \mid \beta < \alpha\} \textit{ is stationary}\}$$

Then $\mathrm{DSS}(\mu^+)$ holds iff $S(\overline{x})$ is stationary.

Proof. We use the equivalence in Theorem 7.1.7 to give a different proof.

First assume $\mathrm{DSS}(\mu^+)$ holds. So there are stationarily many $M \in [H(\mu^+)]^\mu$ which are On-internally unbounded but not On-internally club. Let $C \subseteq \mu^+$ be club and find some $M \prec (H(\mu^+), C, \overline{x}, \in)$ with $M \cap \mu^+ \in \mu^+$ which is On-internally unbounded but not On-internally club. We note $M \cap \mu^+ \in C$. Since M is On-internally unbounded, $M \cap \mu^+$ has cofinality μ. Furthermore, since M is not On-internally club, the set $[M \cap \mu^+]^{<\mu} \smallsetminus ([M \cap \mu^+]^{<\mu} \cap M)$ is stationary in $[M \cap \mu^+]^{<\mu}$. However, by elementarity we clearly have that $[M \cap \mu^+]^{<\mu} \cap M$ equals $\{x_\beta \mid \beta < M \cap \mu^+\}$, so $M \cap \mu^+ \in S(\overline{x}) \cap C$.

On the other hand, assume $S(\overline{x})$ is stationary. It follows that any $M \prec (H(\mu^+), \overline{x}, \in)$ with $M \cap \mu^+ \in S(\overline{x})$ is On-internally unbounded (since $\mathrm{cf}(M \cap \mu^+) = \mu$, so any small subset of $M \cap \mu^+$ is bounded by an ordinal in $M \cap \mu^+$) but not On-internally club (since $[M \cap \mu^+]^{<\mu} \smallsetminus \{x_\beta \mid \beta < M \cap \mu^+\}$ is a stationary subset of $[M \cap \mu^+]^{<\mu}$ which is disjoint from M), so there are in particular stationarily many such M. $\square$

Thus, the easiest way of forcing the negation of DSS is destroying the stationarity of $S((x_\alpha)_{\alpha<\mu^+})$ without adding new $<\mu$-sized subsets to μ^+. This is done straightforwardly by the following forcing:

Definition 8.7.3. Let $\mu^{<\mu} \leq \mu^+$ and $(x_\alpha)_{\alpha<\mu^+}$ enumerate $[\mu^+]^{<\mu}$. Define $\mathbb{P}(\overline{x})$ to consist of all closed bounded subsets p of μ^+ such that whenever $\alpha \in p \cap \operatorname{cof}(\mu)$, $[\alpha]^{<\mu} \setminus \{x_\beta \mid \beta < \alpha\}$ is nonstationary, ordered by end-extension.

Lemma 8.7.4. *Let $\mu^{<\mu} \leq \mu^+$ and $\overline{x} = (x_\alpha)_{\alpha<\mu^+}$ enumerate $[\mu^+]^{<\mu}$.*

1. *$\mathbb{P}(\overline{x})$ is $<\mu$-closed.*
2. *$\mathbb{P}(\overline{x})$ is $<\mu^+$-distributive.*

Proof. For closure, given a descending sequence $(p_\alpha)_{\alpha<\delta}$ for $\delta < \mu$, let $p := \bigcup_{\alpha<\delta} p_\alpha \cup \operatorname{dom}\left(\bigcup_{\alpha<\delta} p_\alpha\right)$ (observing that either the sequence is eventually constant or we have that $\operatorname{cf}\left(\operatorname{dom}\left(\bigcup_{\alpha<\delta} p_\alpha\right)\right) \leq \operatorname{cf}(\delta) < \mu$).

Regarding distributivity, let $\overline{D} = (D_\beta)_{\beta<\mu}$ be a sequence of open dense sets. Choose an increasing and continuous sequence $(N_i)_{i<\mu}$ of elementary substructures of $H(\mu^{++})$ of size $<\mu$ with $N_i \cap \mu \in \mu$ containing $\overline{x}$ and $\overline{D}$ such that $(N_j)_{j<i} \in N_{i+1}$. Let $N := \bigcup_{i<\mu} N_i$. It follows that $N \cap \mu^+ =: \alpha$ is an ordinal and has cofinality μ.

Claim. $\alpha \notin S(\overline{x})$

Proof. Assume toward a contradiction that $[\alpha]^{<\mu} \setminus \{x_\beta \mid \beta < \alpha\}$ is stationary. Clearly $\{N_i \cap \mu^+ \mid i < \mu\}$ is a club subset of $[\alpha]^{<\mu}$. So there exists $i < \mu$ such that $N_i \cap \mu^+ \notin \{x_\beta \mid \beta < \alpha\}$. However, $N_i \cap \mu^+ \in N$, so by elementarity there exists $\beta < \alpha$ with $x_\beta = N_i$, a contradiction. $\square$

Now we can proceed with a standard distributivity argument. Inductively define $(p_i)_{i<\mu}$ so that p_{i+1} is the least refinement of p_i which lies in $N_{i+1} \cap D_i$, using $<\mu$-closure to take limits (since every initial segment of $(N_i)_{i<\mu}$ is in N, every initial segment of $(p_i)_{i<\mu}$ is in N). Then $p := \bigcup_{i<\mu} p_i \cup \{\alpha\}$ is a condition in $\mathbb{P}(\overline{x})$ (since $\alpha \notin S(\overline{x})$) and in every D_i. $\square$

Now it is straightforward to see that $\mathbb{P}(\overline{x})$ forces $\neg\operatorname{DSS}(\mu^+)$: All cardinals below and including μ^+ are preserved and $\overline{x}$ remains an enumeration of $[\mu^+]^{<\mu}$. Lastly, in the extension, $S(\overline{x})$ is nonstationary (the union of a the generic is a club disjoint from $S(\overline{x})$), so $\operatorname{DSS}(\mu^+)$ fails.

The remainder of this section is dedicated to showing that a specific variant of Mitchell forcing forces ISP in such a way that it is not destroyed by further forcing with $\mathbb{P}(\overline{x})$. Fix cardinals $\tau < \mu < \kappa$ such that $\tau^{<\tau} = \tau$ and κ is inaccessible. Also fix any function $l : \kappa \to V_\kappa$.

Definition 8.7.5. Define A to consist of the successor ordinal of inaccessible cardinals in κ. Define $F(l)$ on κ by induction on γ as follows: If γ is inaccessible and $l(\gamma)$ is an $\mathbb{M}(\tau,\mu,\kappa,A,F(l),\gamma)$-name for a $<\mu$-closed forcing, $F(l)(\gamma) = l(\gamma)$. If $\gamma = \delta+2$ is the double successor ordinal of an inaccessible cardinal, $F(l)(\gamma)$ is an $\mathbb{M}(\tau,\mu,\kappa,A,F(l),\gamma)$-name for the forcing

$$\mathbb{P}([\delta]^{<\mu} \cap V[\mathbb{M}(\tau,\mu,\kappa,A,F(l),\delta+1)])$$

Otherwise, $F(l)(\gamma) = \{\check{1}\}$.

Define $\mathbb{M}_9^l(\tau,\mu,\gamma) := \mathbb{M}(\tau,\mu,\kappa,A,F(l),\gamma)$. If $\nu < \kappa$ and G is $\mathbb{M}_9^l(\tau,\mu,\nu)$-generic, define $\mathbb{M}_9^l(G,\tau,\mu,\kappa \smallsetminus \nu) := \mathbb{M}(G,\tau,\mu,\kappa \smallsetminus \nu, A, F(l))$.

We use this specific collapse in order to be able to obtain a "master condition". This is according to the following heuristic: By Theorem 7.1.3, we need a distinction between internal unboundedness and approachability to have $\neg \mathrm{AP}_\mu$. However, $\mathrm{DSS}(\mu^+)$ needs a distinction between internal unboundedness and clubness by Theorem 7.1.2. So, to achieve $\neg \mathrm{AP}_\mu \wedge \neg \mathrm{DSS}(\mu^+)$ we force a distinction between internal clubness and approachability.

By previous arguments, if $\xi = \nu + 1$ for an inaccessible cardinal ν and $\gamma > \xi$, the term ordering on $\mathbb{M}(G,\tau,\mu,\kappa \smallsetminus \xi, A, F, \gamma) * F(l)(\gamma)$ is μ-strategically closed and ξ itself serves as a witness to the existence of some $\xi \in A \smallsetminus \xi$ such that $\mathrm{im}(F \restriction [\xi,\xi]) = \{\check{\mathbb{1}}\}$. Hence as before:

Corollary 8.7.6. *Let $\xi := \nu + 1$ for an inaccessible cardinal ν. Let G be $\mathbb{M}_9^l(\tau,\mu,\xi)$-generic. In $V[G]$, let $\dot{\mathbb{L}}$ be an $\mathbb{M}_9^l(G,\tau,\mu,\kappa \smallsetminus \xi)$-name for a $<\mu$-closed partial order. Then $\mathbb{M}_9^l(\tau,\mu,\kappa \smallsetminus \xi) * \dot{\mathbb{L}}$ has the $<\tau^+$-approximation property.*

And we can show the following:

Theorem 8.7.7. *Let $\tau < \mu < \kappa \leq \lambda$ be cardinals such that $\tau^{<\tau} = \tau$, μ and λ are regular and κ is λ-ineffable. Let l be a λ-ineffable Laver diamond at κ. Then $\mathbb{M}_9^l(\tau,\mu,\kappa) * \mathbb{P}(\overline{x})$ forces $\mathrm{ISP}(\tau^+,\mu^+,\lambda) \wedge \neg \mathrm{DSS}(\mu^+)$.*

Proof. We want to apply Theorem 8.1.2. To this end, let e be a (κ,λ)-list, Θ large, $x \in H(\Theta)$ and $p = (m,\sigma) \in \mathbb{M}_9^l(\tau,\mu,\kappa) * \mathbb{P}(\overline{x})$, where $\overline{x}$ is an $\mathbb{M}_9^l(\tau,\mu,\kappa)$-name for a sequence enumerating all of $([\mu^+]^{<\mu})^{V[\mathbb{M}_9^l(\tau,\mu,\kappa)]}$. Let M be a λ-ineffability witness for κ with respect to e such that

1. $\nu := M \cap \kappa \in \kappa$ is inaccessible
2. $[M \cap \lambda]^{<M \cap \kappa} \subseteq M$
3. $\kappa, \lambda, \tau, \mu, \kappa, \overline{x}, \mathbb{P}(\overline{x}), m, \sigma \in M$
4. $l(\nu) = \pi(\mathbb{P}(\overline{x}))$

We have $m, \sigma \subseteq M$ (because $|m|, |\sigma| < \kappa$) which implies $\pi(m) = m$ and $\pi(\sigma) = \sigma$. Hence we can view p as a condition q in $\mathbb{M}_9^l(\tau,\mu,\nu+1) \cong \mathbb{M}_9^l(\tau,\mu,\nu) * \pi(\mathbb{P}(\overline{x}))$. Let G be $\mathbb{M}_9^l(\tau,\mu,\kappa)$-generic containing q. In $V[G]$, $\pi\colon M \to N$ extends to $\pi\colon M[G] \to N[G']$, where $G' := G \cap \mathbb{M}_9^l(\tau,\mu,\nu)$. Let H' be the $\pi(\mathbb{P}(\overline{x}))^{G'}$-generic filter induced by G.

Claim. *$r := \bigcup H' \cup \{\nu\}$ is a condition in $\mathbb{P}(\overline{x})$ and extends σ^G.*

Proof. Clearly r is a closed bounded subset of κ and $\bigcup H'$ is a sequence of length ν which is unbounded in ν, so we are only left to show that $[\nu]^{<\mu} \smallsetminus \{x_\beta \mid \beta < \nu\}$ is nonstationary in $V[G]$. $\mathbb{M}_9^l(\tau,\mu,\nu+3)$ collapses ν by shooting a club through $[\nu]^{<\mu} \cap V[G']$ (since $\pi(\mathbb{P}(\overline{x}))$ does not add any new $<\mu$-sequences). By Lemma 5.3.5 this is equal to $[\nu]^{<\mu} \cap N[G']$ and by elementarity, $[\nu]^{<\mu} \cap N[G'] = \{x_\beta \mid \beta < \nu\}$. This implies that $[\nu]^{<\mu} \setminus \{x_\beta \mid \beta < \nu\}$ is nonstationary in $V[G]$.

r extends σ^G because $\sigma^G = \pi(\sigma)^{G'} = \sigma^{G'}$ and so $\sigma^G \in H'$. $\square$

If H is $\mathbb{P}(\overline{x})^G$-generic over $V[G]$, $\pi[H]$ is equal to the $\pi(\mathbb{P}(\overline{x}))^{G'}$-generic filter induced by G. Ergo $\pi[G * H \cap M]$ is equivalent to the $\mathbb{M}_9^l(\tau,\mu,\nu+1)$-generic filter induced by G (in particular, it is $\pi(\mathbb{M}_9^l(\tau,\mu,\kappa) * \mathbb{P}(\overline{x}))$-generic over V)and the pair $(V[\pi[G * H \cap M]], V[G * H])$ has the $<\tau^+$-approximation property by Lemma 8.7.6.

Ergo $q \in \mathbb{M}_9^l(\tau,\mu,\kappa)$ forces the existence of a condition $\dot{r}$ which forces the desired statement. In summary, $(q,\dot{r}) \leq (m,\sigma)$ is as required. By Lemma 8.7.4, $\mathbb{M}_9^l(\tau,\mu,\kappa)$ forces $\mathbb{P}(\overline{x})$ to be $<\kappa$-distributive. So by Theorem 8.1.2, $\mathbb{M}_9^l(\tau,\mu,\kappa) * \mathbb{P}(\overline{x})$ forces $\mathrm{ISP}(\tau^+,\kappa,\lambda)$. As $\mathbb{P}(\overline{x})$ forces $\neg\,\mathrm{DSS}(\mu^+)$, we are done. $\square$

8.8 ISP and Club Stationary Reflection

Gilton, Levine and Stejskalová showed in [GLS23] that $\mathrm{TP}(\tau^{++})$ is compatible together with club stationary reflection at τ^{++} (relative to a weakly compact cardinal). In this section we will adapt their techniques to show that $\mathrm{ISP}(\tau^+,\mu^+,\lambda)$ is also compatible with club stationary reflection at μ^+.

Definition 8.8.1. Let μ be a regular cardinal. *Club Stationary Reflection*, denoted $\mathrm{CSR}(\mu^+)$, holds at μ^+ if for every stationary set $S \subseteq \mu^+ \cap \mathrm{cof}(<\mu)$ there is a club $C \subseteq \mu^+$ such that whenever $\alpha \in C$ has cofinality μ , $S \cap \alpha$ is stationary in α.

Classically, $\mathrm{CSR}(\mu^+)$ is obtained by adding many clubs to μ^+:

Definition 8.8.2. Let δ be a cardinal and $X \subseteq \delta$. The poset $\mathrm{CU}(X)$ consists of closed bounded subsets $c \subseteq X$, ordered by end-extension.

One checks easily that if X is unbounded, $\mathrm{CU}(X)$ adds a club which is contained in X (depending on the structure of X, it might however collapse cardinals).

Definition 8.8.3. Let $\delta = \delta^{<\delta}$ be a cardinal. $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha)_{\alpha<\delta^+}$ is a *standard club-adding iteration of length* δ^+ if $\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha = \mathrm{CU}(\dot{X}) \wedge \dot{X} \subseteq \check{\delta}$ and the iteration has $<\delta$-support.

If $\delta^{<\delta} = \delta$, then $|\mathrm{CU}(X)| = \delta$. Additionally:

Lemma 8.8.4. *Let $\mathbb{P} := (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha)_{\alpha<\delta^+}$ be a standard club-adding iteration of length δ^+. For any $\alpha < \delta^+$, if $\mathbb{P}_\alpha$ is $<\delta$-distributive and has a dense subset of size δ, $\mathbb{P}_{\alpha+1}$ also contains a dense subset of size δ.*

Proof. The set simply consists of $(p,\check{c}) \in \mathbb{P}_{\alpha+1}$, where p is in the small dense subset of $\mathbb{P}_\alpha$ and c is a (true) closed bounded subset of δ (with $p \Vdash \check{c} \in \dot{\mathbb{Q}}_\alpha$). $\square$

Now assume $\mathbb{P}$ is a standard club-adding iteration of length δ such that each $\mathbb{P}_\alpha$ for $\alpha < \delta^+$ is $<\delta$-distributive and $\delta^{<\delta} = \delta$. Then inductively each $\mathbb{P}_\alpha$ contains a dense subset of size δ for $\alpha < \delta^+$. In particular $\mathbb{P}_{\delta^+}$ is δ^+-c.c. by an easy application of the Δ-System-Lemma and in this case any function $f\colon \delta \to V$ added by $\mathbb{P}_{\delta^+}$ has been added by some $\mathbb{P}_\alpha$ for $\alpha < \delta^+$.

Now we define a variant of Mitchell forcing which makes ISP indestructible under forcing with any standard club-adding iteration. Fix, for the last time, regular cardinals $\tau < \mu < \kappa$ such that $\tau^{<\tau} = \tau$ and κ is inaccessible. Also fix a function $l\colon \kappa \to V_\kappa$.

Definition 8.8.5. Let A consist of the successors of regular cardinals in κ and define $F(l)$ by induction on γ. If γ is an inaccessible cardinal and $l(\gamma)$ is an $\mathbb{M}(\tau,\mu,\kappa,A,F,\gamma)$-name for a $<\mu$-closed partial order, let $F(l)(\gamma) := l(\gamma)$. If $\gamma = \delta + 2$ for an inaccessible cardinal δ, let $F(l)(\gamma)$ be an $\mathbb{M}(\tau,\mu,\kappa,A,F,\gamma)$-name for $\mathrm{Coll}(\check{\mu},\check{\delta})$.

Define $\mathbb{M}^l_{10}(\tau,\mu,\gamma) := \mathbb{M}(\tau,\mu,\kappa,A,F(l),\gamma)$ and for $\xi < \kappa$ and an $\mathbb{M}^l_{10}(\tau,\mu,\xi)$-generic filter G, define $\mathbb{M}^l_{10}(G,\tau,\mu,\kappa\setminus\xi) := \mathbb{M}(G,\tau,\mu,\kappa\setminus\xi,A,F(l))$.

As before, we have the following:

Lemma 8.8.6. *Let $\xi := \nu+1$ for an inaccessible cardinal ν. Let G be $\mathbb{M}^l_{10}(\tau,\mu,\xi)$-generic. In $V[G]$, let $\dot{\mathbb{L}}$ be an $\mathbb{M}^l_{10}(G,\tau,\mu,\kappa\setminus\xi)$-name for a $<\mu$-closed partial order. Then $\mathbb{M}^l_{10}(\tau,\mu,\kappa\setminus\xi) * \dot{\mathbb{L}}$ has the $<\tau^+$-approximation property.*

Now we work in an extension $V[G]$ by $\mathbb{M}^l_{10}(\tau,\mu,\kappa)$ (where clearly $\kappa = \mu^+$) Suppose that $2^\kappa = \kappa^+$ (which can be achieved by preparing the ground model). We define a standard club-adding iteration that forces $\mathrm{CSR}(\mu^+)$. To this end, let $F\colon \kappa^+ \to \kappa^+ \times \kappa^+$ be a bijection such that if $F(\alpha) = (\beta,\gamma)$, $\beta \leq \alpha$. By induction on α we define an iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha)_{\alpha<\kappa^+}$ as well as sequences $(\dot{S}_\alpha)_{\alpha<\kappa^+}$ and $(\dot{T}_\alpha)_{\alpha<\kappa^+}$. Assume all objects have been defined until α and let $F(\alpha) = (\beta,\gamma)$. If α is a limit ordinal, let $\mathbb{P}_\alpha$ be the $<\kappa$-supported limit of the iteration constructed thus far. Otherwise, let $\mathbb{P}_\alpha := \mathbb{P}_{\alpha-1} * \dot{\mathbb{Q}}_{\alpha-1}$. Let $\dot{S}_\alpha$ be the γth nice $\mathbb{P}_\beta$-name for a stationary subset of $\kappa \cap \mathrm{cof}(<\mu)$ and let $\dot{T}_\alpha$ be a $\mathbb{P}_\beta$-name for the set of all points ρ such that either ρ has cofinality $<\mu$ or $\dot{S}_\alpha$ reflects in ρ. Let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathrm{CU}(\dot{T}_\alpha)$.

We will later show (and thus assume now) that each $\mathbb{P}_\alpha$ for $\alpha < \kappa^+$ is $<\kappa$-distributive. Ergo $\mathbb{P}_{\kappa^+}$ has the κ^+-c.c. Additionally:

Lemma 8.8.7. *For all $\alpha \leq \kappa^+$, $\mathbb{P}_\alpha$ is $<\mu$-closed.*

Proof. By induction, it suffices to show that for all α, $\mathbb{P}_\alpha$ forces $\dot{\mathbb{Q}}_\alpha$ to be $<\mu$-closed. However, this is clear as we shoot clubs into sets containing every point of cofinality $<\mu$. □

Now we have to consider two cases: If we are only concerned with $\mathrm{ISP}(\tau^+,\mu^+,\mu^+)$, any supposed counterexample (which has size μ^+) has been added by $\mathbb{M} * \mathbb{P}_\alpha$ for some $\alpha < \mu^{++}$. However, if we want to have $\mathrm{ISP}(\tau^+,\mu^+,\lambda)$ for $\lambda \geq \mu^{++}$, it is possible that the iteration has only added the counterexample at the final stage. Luckily, in that case we also have a stronger assumption of λ-ineffability which allows a similar argument to be carried out.

Theorem 8.8.8. *Assume $\lambda = \lambda^{<\kappa}$ is regular, κ is λ-ineffable and l is a λ-ineffable Laver diamond at κ. Then $\mathbb{M}(\tau, \mu, \kappa) * \mathbb{P}_{\kappa^+}$ forces $\mathrm{ISP}(\tau^+, \mu^+, \lambda) \wedge \mathrm{CSR}(\mu^+)$.*

Proof. We do the proof simultaneously for the case $\lambda = \kappa$ and $\lambda \geq \kappa^+$. If $\lambda = \kappa$, let $\alpha < \kappa^+$ be arbitrary (it suffices to show that $\mathbb{M}^l_{10}(\tau, \mu, \kappa) * \mathbb{P}_\alpha$ satisfies the assumptions of Theorem 8.1.2 for all $\alpha < \kappa^+$). Otherwise, let $\alpha := \kappa^+$.

Let K be a transitive model of enough ZFC (i.e. enough to be able to define $\Vdash$) containing $\mathbb{M}^l_{10}(\tau, \mu, \kappa) * \mathbb{P}_\alpha$, α and every $\dot{S}_\beta$ for $\beta < \alpha$. Let F be a bijection between K and either κ or κ^+ depending on the case and let $A := F[\in \restriction (K \times K)]$.

We want to apply Theorem 8.1.2. Let Θ be large, $x \in H(\Theta)$, $(m, \sigma) \in \mathbb{M}^l_{10}(\tau, \mu, \kappa) * \mathbb{P}_\alpha$ and e a (κ, λ)-list. Using Corollary 5.2.7, find a λ-ineffability witness M for κ with respect to e such that the following holds:

1. M is Π^1_1-correct about λ with respect to A.
2. $x, m, \sigma, \tau, \mu, \kappa, \mathbb{P}_\alpha, Z, F \in M$,
3. $\nu := M \cap \kappa$ is inaccessible,
4. $[M \cap \lambda]^{<M \cap \kappa} \subseteq M$,
5. $l(\nu) = \pi(\mathbb{P}_\alpha)$.

As before $m, \sigma \subseteq M$, so we can imagine $(m, \sigma) \in \mathbb{M}^l_{10}(\tau, \mu, \nu) * \pi(\mathbb{P}_\alpha)$ as a condition q in $\mathbb{M}^l_{10}(\tau, \mu, \nu + 1)$. Let G be $\mathbb{M}^l_{10}(\tau, \mu, \kappa)$-generic containing q.

Claim. *There exists a condition $r \leq \sigma^G$ such that whenever K is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V containing r, $\pi[K \cap M]$ is $\pi(\mathbb{P} * \dot{\mathbb{Q}})$-generic over V and $(V[\pi[K]], V[K])$ has the $<\pi(\delta)$-approximation property.*

Proof. Let $G' * H'$ be the $\mathbb{M}^l_{10}(\tau, \mu, \nu) * \pi(\mathbb{P}_\alpha)$-generic filter induced by G, $H' = (K_\beta)_{\beta \leq \pi(\alpha)}$ (where each K_β is $\mathbb{P}_\beta$-generic). For each $\beta < \pi(\alpha)$, let $C_\beta := \bigcup L_\beta$ (where L_β is $\dot{\mathbb{Q}}^{K_\beta}_\beta$-generic) which is club in ν. We define r as follows: For each $\beta \in \alpha \cap M$, $r(\beta) = C_{\pi(\beta)} \cup \{\nu\}$. As each $C_{\pi(\beta)}$ is club in ν, each $r(\beta)$ is a closed subset of κ (and $\dot{S}_\beta$ is forced to reflect at any $\gamma \in r(\beta) \cap \mathrm{cof}(<\mu)$ by elementarity), so we are done after showing that ν is forced to be a reflection point of each $\dot{S}_\beta$, $\beta \in M$. We will prove this by induction. We note that r extends σ^G because $\sigma^G = \pi(\sigma^G) = \pi(\sigma)^{G'} = \sigma^{G'}$ and ergo σ^G is in the induced filter.

Assume the statement holds for all $\gamma < \beta$. If $\beta \notin M$, we are done, so assume $\beta \in M$.

Let H be $\mathbb{P}_\beta$-generic containing $r \restriction \beta$. By the inductive hypothesis, $\pi[G * H] = G' * K_{\pi(\beta)}$. By Lemma 5.3.4 and Lemma 5.3.6, $\pi\colon M \to N$ extends to $\pi\colon M[G * H] \to N[G' * K_{\pi(\beta)}]$ by setting $\pi(\sigma^{G*H}) = \pi(\sigma)^{G' * K_{\pi(\beta)}}$. Thus

$$\dot{S}^{G*H}_\beta \cap \nu = \pi(\dot{S}^{G*H}_\beta) = \pi(\dot{S}_\beta)^{G' * K_{\pi(\beta)}} =: S'_\beta$$

Now we show that S'_β is stationary in ν in $V[G' * K'_\beta]$. Otherwise there would be an $\mathbb{M}^l_{10}(\tau,\mu,\nu) * \pi(\mathbb{P}_\beta)$-name $\dot{C}$ and a condition $\pi(p) \in G' * K'_\beta = \pi[G * H]$ forcing $\dot{C}$ to be club in ν and to be disjoint from $\pi(\dot{S}_\beta)$. Let $\phi[r,\xi,\tau,\mathbb{R}]$ be a first-order sentence in the language $\mathcal{L}(P_0, P_1)$, where P_0 is a binary and P_1 a unary predicate, stating that P_1 is an $\mathbb{R}$-name forced by $r \in \mathbb{R}$ to be club in ξ and disjoint from τ. Since $\pi(K)$ is a transitive model of enough ZFC containing all relevant parameters it follows that

$$(\pi(K), \in, \dot{C}) \models \phi[\pi(p), \pi(\kappa), \pi(\dot{S}_\beta), \mathbb{M}^l_{10}(\tau,\mu,\nu) * \pi(\mathbb{P}_\beta)]$$

because that formula is sufficiently elementary. Since $\pi(F)$ is a bijection between $\pi(K)$ and $\pi(\lambda)$ (assume for simplicity that λ is either κ or κ^+ since the model only gets stronger as we increase λ) we know that

$$(\pi(\lambda), \pi(A), \pi(F)[\dot{C}]) \models \phi[\pi(F)(\pi(p)), \pi(F)(\pi(\kappa)), \pi(F)(\pi(\dot{S}_\beta)), \pi(F)(\mathbb{M}^l_{10}(\tau,\mu,\nu) * \pi(\mathbb{P}_\beta))] \tag{*}$$

and so we can assume that there is some $X \in N$ such that $(*)$ holds when X is substituted for $\pi(F)[\dot{C}]$. However, $\pi(F)^{-1}[X]$ is in N by elementarity and since

$$(\pi(K), \in, \pi(F)^{-1}[X]) \models \phi[\pi(p), \pi(\kappa), \pi(\dot{S}_\beta), \mathbb{M}^l_{10}(\tau,\mu,\nu) * \pi(\mathbb{P}_\beta)]$$

we know that $\pi(F)^{-1}[X]$ is an $\mathbb{M}^l_{10}(\tau,\mu,\nu) * \mathbb{P}_\beta$-name forced by $\pi(p)$ to be club in $\pi(\kappa)$ and disjoint from $\pi(\dot{S}_\beta)$. This of course also holds in N because the sentence is sufficiently elementary. Ergo, as π^{-1} is an elementary embedding from N into some $H(\Theta)$, $\pi^{-1}(\pi(F)^{-1}[X])$ is an $\mathbb{M}^l_{10}(\tau,\mu,\kappa) * \pi(\mathbb{P}_\beta)$-name forced by p to be club in κ and disjoint from $\dot{S}_\beta$, a clear contradiction.

Lastly, we show that this stationarity is preserved when going from $V[G' * K'_\beta]$ to $V[G * H]$. In $V[G']$ and thus in $V[G' * K'_\beta]$, $\nu = \mu^+$ and S'_β is a stationary subset of $\nu \cap \operatorname{cof}(<\mu)$. Since $V[G' * K_{\pi(\alpha)}]$ is a $<\mu$-closed extension of $V[G' * K'_\beta]$, S'_β remains a stationary subset of $\nu \cap \operatorname{cof}(<\mu)$ in that extension by Lemma 2.4.7. In $V[G' * K'_\beta]$, we either still have $\nu = \mu^+$ or $|\nu| = \mu$. In either case, the stationarity of S'_β is preserved when going from $V[G' * K_{\pi(\alpha)}]$ to $V[G]$, since that is an extension by a forcing which can be projected onto from the product of a τ^+-c.c. and a $<\mu$-closed poset. Lastly, in $V[G]$, we definitely have $|\nu| = \mu$ and again this implies that the stationarity of S'_β is preserved by the $<\mu$-closed extension to $V[G * H]$. □

Ergo q forces the existence of a condition $\dot{r}$ which forces the desired statement, so $(q, \dot{r})$ (which extends (m, σ)) is as required. The last thing to prove is that $\mathbb{P}_\alpha$ is forced to be $<\kappa$-distributive. Recall that α was either some arbitrary ordinal below κ^+ or equal to κ^+ depending on the desired degree of ISP (and the ineffability of κ).

Claim. *If $\dot{f} \in M$ is an $\mathbb{M}^l_{10}(\tau,\mu,\kappa) * \mathbb{P}_\alpha$-name for a β-sequence of ordinals, $\beta < \nu$, $\dot{f}$ is in $V[G]$.*

Proof. Let σ be the isomorphism between $\mathbb{M}^l_{10}(\tau,\mu,\nu) * \pi(\mathbb{P}_\alpha)$ and $\mathbb{M}^l_{10}(\tau,\mu,\nu+1)$. In $V[G * H]$, $\pi(\dot{f}^{G*H}) = (\pi(\dot{f}))^{G' * K'_\beta} = (\sigma_*(\pi(\dot{f})))^{G \cap \mathbb{M}^l_{10}(\tau,\mu,\nu+1)}$, so $\pi(\dot{f}^{G*H}) \in V[G]$. However, because $\beta < \nu$, $\dot{f}^{G*H} \subseteq M[G * H]$, so $\pi(\dot{f}^{G*H}) = \pi[\dot{f}^{G*H}]$ and $\dot{f}^{G*H} = \pi^{-1}[\pi(\dot{f}^{G*H})] \in V[G]$. □

Which implies:

Claim. $\mathbb{M}^l_{10}(\tau, \mu, \kappa)$ *forces* $\mathbb{P}_\alpha$ *to be* $<\kappa$*-distributive.*

Proof. For any $\dot{f}$ which is forced to have domain $\beta < \kappa$ we can find a λ-ineffability witness M for κ with respect to e as required above such that $\dot{f} \in M$ and thus $M \cap \kappa \geq \beta$. $\square$

Ergo $\mathbb{M}^l_{10}(\tau, \mu, \kappa) * \mathbb{P}_\alpha$ satisfies the conditions of Theorem 8.1.2 and forces $\mathrm{ISP}(\tau^+, \kappa, \lambda)$.

The last thing to show is that $\mathbb{P}_{\kappa^+}$ forces $\mathrm{CSR}(\mu^+)$. If $\dot{S}$ is a $\mathbb{P}_{\kappa^+}$-name for a stationary subset of $\kappa \cap \mathrm{cof}(<\mu)$, then we can assume $\dot{S}$ is a $\mathbb{P}_\beta$-name for some $\beta < \kappa^+$ and in particular the γth nice $\mathbb{P}_\beta$-name for a stationary subset of $\kappa \cap \mathrm{cof}(<\mu)$. This implies that at stage $\alpha = F^{-1}(\beta, \gamma)$ we have added a club of reflection points for $\dot{S}$ and this is still true after forcing with the rest of $\mathbb{P}_{\kappa^+}$ due to its distributivity.

$\square$

CHAPTER 9

Conclusion and Open Questions

In this thesis, we have answered a number of open questions and introduced new techniques to tackle existing ones, opening up potential future research directions. In this chapter, we give some new (and some well-known) interesting open questions:

9.1 Strong Distributivity

Since the concept of strong distributivity is new, there are several technical questions which have not yet been answered.

Recall that we showed in Lemma 3.4.11 that under Martin's Maximum, any $<\omega_1$-distributive forcing notion which preserves stationary subsets of ω_1 is strongly $<\omega_1$-distributive. This leaves open the following two questions:

Question 9.1.1. Is it consistent for $\kappa > \omega_1$ that any $<\kappa$-distributive forcing notion which preserves stationary subsets of κ is strongly $<\kappa$-distributive?

Question 9.1.2. What is the consistency strength of the assertation that any $<\omega_1$-distributive forcing notion which preserves stationary subsets of ω_1 is strongly $<\omega_1$-distributive?

We note that the assertion implies that there do not exist Suslin trees, since a Suslin tree is countably distributive and c.c.c., so it preserves stationary subsets of ω_1 but it cannot be strongly $<\omega_1$-distributive. It also implies that any stationary subset of $E_\omega^\delta := \{\alpha \in \delta \mid \mathrm{cf}(\alpha) = \omega\}$ contains a closed copy of ω_1, if $\mathrm{cf}(\delta) \geq \omega_2$, so its consistency strength is at least that of a Mahlo cardinal.

We have also showed that consistently strong distributivity, just like normal distributivity, is not productive. But we do not know if that is always the case:

Question 9.1.3. Is it consistent for one or for all κ that the product of two strongly $<\kappa$-distributive posets is strongly $<\kappa$-distributive?

9.2 Variants of Internal Approachability

Most of these questions were asked previously by Krueger in [Kru09]. We give them here again and some possible avenues towards solutions.

Question 9.2.1. What is the consistency strength of the assertion that the properties of being internally unbounded and internally stationary are distinct for stationarily many $[H(\Theta)]^{\mu}$?

We have shown that for the other two inclusions (internally stationary $\subseteq$ internally club $\subseteq$ internally approachable) a distinction can be forced for $[H(\Theta)]^{\mu}$ from a Mahlo cardinal. In [FK07], Friedman and Krueger force the existence of a disjoint club sequence on ω_2 (and thus a distinction between internal unboundedness and stationarity in $[H(\omega_2)]^{\omega_1}$) from a Mahlo cardinal. It is highly likely that in the same model the distinction holds for stationarily many $N \in [H(\Theta)]^{\aleph_1}$ for arbitrary $H(\Theta)$.

Question 9.2.2. Can there be a disjoint stationary sequence on ω_3 while $2^{\omega} = \omega_1$? Can there be a disjoint club sequence on ω_3?

Both questions are related to the problem of finding a general technique to obtain models in which $\mathrm{Add}(\omega_1)$ forces that there are stationarily many new sets in $[\omega_2]^{\omega_1}$ which are internally approachable of length ω_1. Additionally, finding a technique to obtain the result for larger $\mathrm{Add}(\mu)$ could open up a path towards a solution to the following problem:

Question 9.2.3. Can there be a model in which the properties of being internally stationary and internally club are distinct for stationarily many $N \in [H(\Theta)]^{\omega_{n+1}}$, $n \in \omega$ arbitrary, such that $2^{\omega_n} = \omega_{n+2}$ for every $n \in \omega$?

A major set-theoretic problem is the question if it is possible to have the tree property at every regular cardinal. We ask here for another global result, although one that is expectedly easier to obtain then the global tree property, since the property is more easily preserved and cannot occur at successors of singular cardinals.

Question 9.2.4. Is it consistent that for every regular cardinal μ and every $\Theta > \mu$, the properties of being internally club and internally approachable are distinct for stationarily many $N \in [H(\Theta)]^{\mu}$?

It is possible that combining Radin forcing with our special variant of Mitchell Forcing from Section 7.3 and the techniques in [Ung17] could yield an answer to the above question.

Question 9.2.5. Is it consistent that $\mu^+ \notin I[\mu^+]$ and there is a club $C \subseteq [H(\mu^+)]^{\mu}$ such that any $M \in C$ is either On-internally approachable or not On-internally unbounded?

By Theorem 7.1.8 this would require $2^{<\mu} > \mu^+$. A similar cardinal arithmetic was used by Mitchell in [Mit09] to obtain the consistency of the statement "No stationary $S \subseteq \omega_2 \cap \mathrm{cof}(\omega_1)$ is in $I[\omega_2]$" and by Krueger in [Kru19b] to obtain the consistency of the statement "$I[\omega_2]$ does not have a maximal set modulo clubs". It seems highly likely that these models have a connection to the question above.

9.3 The Ineffable Slender Property

For the tree property, a model was constructed by Cummings and Foreman in [CF98] where every $\aleph_{n+2}$ has the tree property. Fontanella showed in her PhD thesis that, in the same model, every $\aleph_{n+2}$ even has the super tree property. However, these techniques cannot be adapted to show that ISP holds. In Section 8.6, we showed that it is possible to make ISP indestructible under $<\kappa$-directed closed forcing which is a major step in the proof by Cummings and Foreman. However, it is unclear if that can be adapted to obtain an answer to the following question (which was previously asked by Mohammadpour in [Moh23]):

Question 9.3.1. Is it consistent that $\mathrm{ISP}(\omega_1, \aleph_{n+2}, \geq \aleph_{n+2})$ holds for every $n \in \omega$?

A different possible pattern is the following:

Question 9.3.2. Is it consistent that $\mathrm{ISP}(\omega_{n+2}, \omega_{n+2}, \geq \omega_{n+2})$ together with $2^{\omega_n} = \omega_{n+2}$ holds for every $n \in \omega$?

Bibliography

[AS83] Uri Abraham and Saharon Shelah. "Forcing Closed Unbounded Sets". In: *The Journal of Symbolic Logic* 48.3 (1983), pp. 643–657. ISSN: 00224812. URL: http://www.jstor.org/stable/2273456.

[CF98] James Cummings and Matthew D. Foreman. "The Tree Property". In: *Advances in Mathematics* 133 (1998), pp. 1–32. URL: https://api.semanticscholar.org/CorpusID:5664759.

[CK16] Sean Cox and John Krueger. "Quotients of Strongly Proper Forcings and Guessing Models". In: *The Journal of Symbolic Logic* 81.1 (2016), pp. 264–283. ISSN: 00224812, 19435886. URL: http://www.jstor.org/stable/43864295.

[Cox21] Sean D. Cox. "Forcing Axioms, Approachability, and Stationary Set Reflection". In: *The Journal of Symbolic Logic* 86.2 (2021), pp. 499–530.

[Cum+18] James Cummings et al. "The Eightfold Way". In: *The Journal of Symbolic Logic* 83.1 (2018), pp. 349–371. ISSN: 00224812, 19435886. URL: https://www.jstor.org/stable/26600322.

[DPZ80] Carlos Di Prasco and William Zwicker. "Flipping properties and supercompact cardinals". eng. In: *Fundamenta Mathematicae* 109.1 (1980), pp. 31–36. URL: http://eudml.org/doc/211167.

[FJ98] Qi Feng and Thomas Jech. "Projective Stationary Sets and a Strong Reflection Principle". In: *Journal of the London Mathematical Society* 58.2 (1998), pp. 271–283. eprint: https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/S0024610798006462. URL: https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/S0024610798006462.

[FK07] Sy-David Friedman and John Krueger. "Thin Stationary Sets and Disjoint Club Sequences". In: *Transactions of the American Mathematical Society* 359.5 (2007), pp. 2407–2420. ISSN: 00029947. URL: http://www.jstor.org/stable/20161681.

[For05] Matthew Foreman. "Some Problems in Singular Cardinals Combinatorics". In: *Notre Dame Journal of Formal Logic* 46.3 (2005), pp. 309 –322. URL: https://doi.org/10.1305/ndjfl/1125409329.

[For83] Matthew Foreman. "Games Played on Boolean Algebras". In: *The Journal of Symbolic Logic* 48.3 (1983), pp. 714–723. URL: http://www.jstor.org/stable/2273464.

[FT05] Matthew Foreman and Stevo Todorcevic. "A New Löwenheim-Skolem Theorem". In: *Transactions of the American Mathematical Society* 357.5 (2005), pp. 1693–1715. ISSN: 00029947. URL: http://www.jstor.org/stable/3845130.

[Git85] Moti Gitik. "Nonsplitting Subset of $P_\kappa(\kappa +)$". In: *The Journal of Symbolic Logic* 50.4 (1985), pp. 881–894. URL: http://www.jstor.org/stable/2273978.

[GK09] Moti Gitik and John Krueger. “Approachability at the Second Successor of a Singular Cardinal”. In: *The Journal of Symbolic Logic* 74.4 (2009), pp. 1211–1224. ISSN: 00224812, 19435886. URL: http://www.jstor.org/stable/40378264.

[GLS23] Thomas Gilton, Maxwell Levine, and Šárka Stejskalová. “Trees and Stationary Reflection at Double Successors of Regular Cardinals”. In: *The Journal of Symbolic Logic* 88.2 (2023), pp. 780–810.

[Ham02] Joel David Hamkins. “A class of strong diamond principles”. 2002. arXiv: math/0211419 [math.LO].

[Ham03] Joel David Hamkins. “Extensions with the approximation and cover properties have no new large cardinals”. eng. In: *Fundamenta Mathematicae* 180.3 (2003), pp. 257–277. URL: http://eudml.org/doc/286591.

[Ham98] Joel David Hamkins. “The Lottery Preparation”. In: *Ann. Pure Appl. Log.* 101 (1998), pp. 103–146. URL: https://api.semanticscholar.org/CorpusID:15579965.

[Ham99] Joel David Hamkins. “Gap Forcing: Generalizing the Lévy-Solovay Theorem”. In: *The Bulletin of Symbolic Logic* 5.2 (1999), pp. 264–272. ISSN: 10798986. URL: http://www.jstor.org/stable/421092.

[HLN19] Peter Holy, Philipp Lücke, and Ana Njegomir. “Small embedding characterizations for large cardinals”. In: *Annals of Pure and Applied Logic* 170.2 (2019), pp. 251–271. URL: https://www.sciencedirect.com/science/article/pii/S0168007218301167.

[IY02] Tetsuya Ishiu and Yasuo Yoshinobu. “Directive Trees and Games on Posets”. In: *Proceedings of the American Mathematical Society* 130.5 (2002), pp. 1477–1485. URL: http://www.jstor.org/stable/2699609.

[Jak23] Hannes Jakob. “Disjoint Stationary Sequences on an Interval of Cardinals”. 2023. arXiv: 2309.01986 [math.LO].

[Jak24a] Hannes Jakob. “Cascading Variants of Internal Approachability”. 2024. arXiv: 2404.18571 [math.LO]. URL: https://arxiv.org/abs/2404.18571.

[Jak24b] Hannes Jakob. “Slender Trees and the Approximation Property”. 2024. arXiv: 2401.13446 [math.LO].

[Jec03] Thomas Jech. *Set Theory: The Third Millennium Edition*. Springer, 2003.

[Jec73] Thomas J. Jech. “Some combinatorial problems concerning uncountable cardinals”. In: *Annals of Mathematical Logic* 5.3 (1973), pp. 165–198. ISSN: 0003-4843. URL: https://www.sciencedirect.com/science/article/pii/0003484373900144.

[JL25] Hannes Jakob and Maxwell Levine. “Distinguishing Internally Club and Approachable on an Infinite Interval”. In: *Bulletin of the London Mathematical Society* (2025), to appear.

[Kru05] John Krueger. "Destroying Stationary Sets". In: *Israel Journal of Mathematics* 147 (2005), pp. 285–328. URL: https://doi.org/10.1007/BF02785369.

[Kru07] John Krueger. "Internally club and approachable". In: *Advances in Mathematics* 213.2 (2007), pp. 734–740. ISSN: 0001-8708. URL: https://www.sciencedirect.com/science/article/pii/S0001870807000291.

[Kru08a] John Krueger. "A General Mitchell Style Iteration". In: *Mathematical Logic Quarterly* 54.6 (2008), pp. 641–651.

[Kru08b] John Krueger. "Internal Approachability and Reflection". In: *Journal of Mathematical Logic* 08.01 (2008), pp. 23–39. eprint: https://doi.org/10.1142/S0219061308000701. URL: https://doi.org/10.1142/S0219061308000701.

[Kru08c] John Krueger. "Internally club and approachable for larger structures". eng. In: *Fundamenta Mathematicae* 201.2 (2008), pp. 115–129. URL: http://eudml.org/doc/283095.

[Kru09] John Krueger. "Some applications of mixed support iterations". In: *Annals of Pure and Applied Logic* 158.1 (2009), pp. 40–57. URL: https://www.sciencedirect.com/science/article/pii/S0168007208001486.

[Kru19a] John Krueger. "Guessing models imply the singular cardinal hypothesis". In: *Proceedings of the American Mathematical Society* 147 (2019), pp. 5427–5434. URL: https://www.ams.org/journals/proc/2019-147-12/S0002-9939-2019-14739-0/.

[Kru19b] John Krueger. "The Approachability Ideal Without a Maximal Set". In: *Annals of Pure and Applied Logic* 170.3 (2019), pp. 297–382.

[Kun11] K. Kunen. *Set Theory*. Studies in Logic: Mathematical. College Publications, 2011. URL: https://books.google.de/books?id=Zn8ppwAACAAJ.

[Lav78] Richard Laver. "Making the supercompactness of κ indestructible under κ-directed closed forcing". In: *Israel Journal of Mathematics* 29 (1978), pp. 385–388. URL: https://doi.org/10.1007/BF02761175.

[Lev24] Maxwell Levine. "On disjoint stationary sequences". In: *Pacific Journal of Mathematics* 332.1 (2024), pp. 147–165.

[LHS24a] Chris Lambie-Hanson and Šárka Stejskalová. "Guessing Models, Trees, and Cardinal Arithmetic". In: *Israel Journal of Mathematics* (2024), to appear.

[LHS24b] Chris Lambie-Hanson and Šárka Stejskalová. "Strong tree properties, Kurepa trees, and guessing models". In: *Monatshefte für Mathematik* 203 (2024), pp. 111–148. URL: https://doi.org/10.1007/s00605-023-01922-2.

[Lie] Andreas Lietz. *Are the completeness Games $G_{\lambda+1}(P)$ and $G_{\lambda^+}(P)$ equivalent for INC?* MathOverflow. eprint: https://mathoverflow.net/q/442582. URL: https://mathoverflow.net/q/442582.

[Mag74] M. Magidor. "Combinatorial Characterization of Supercompact Cardinals". In: *Proceedings of the American Mathematical Society* 42.1 (1974), pp. 279–285. ISSN: 00029939, 10886826. URL: http://www.jstor.org/stable/2039718.

[Men75] Telis K. Menas. "On strong compactness and supercompactness". In: *Annals of Mathematical Logic* 7.4 (1975), pp. 327–359. ISSN: 0003-4843. URL: https://www.sciencedirect.com/science/article/pii/0003484375900091.

[Mit06] William J. Mitchell. "On the Hamkins Approximation Property". In: *Annals of Pure and Applied Logic* 144.1-3 (2006), pp. 126–129.

[Mit09] William J. Mitchell. "$I[\omega_2]$ can be the nonstationary ideal on $\omega_2 \cap \mathrm{cof}(\omega_1)$". In: *Transactions of the American Mathematical Society* 361 (2009), pp. 561–601. URL: https://www.ams.org/journals/tran/2009-361-02/S0002-9947-08-04664-3/?active=current.

[Mit72] William Mitchell. "Aronszajn trees and the independence of the transfer property". In: *Annals of Mathematical Logic* 5.1 (1972), pp. 21–46. ISSN: 0003-4843. URL: https://www.sciencedirect.com/science/article/pii/0003484372900174.

[MKS64] J. Mayer-Kalkschmidt and E. Steiner. "Some theorems in set theory and applications in the ideal theory of partially ordered sets". In: *Duke Mathematical Journal* 31.2 (1964), pp. 287 –289. URL: https://doi.org/10.1215/S0012-7094-64-03127-8.

[Moh23] Rahman Mohammadpour. "A Road To Compactness Through Guessing Models". 2023. arXiv: 2210.02514 [math.LO].

[MV21] Rahman Mohammadpour and Boban Veličković. "Guessing models and the approachability ideal". In: *Journal of Mathematical Logic* 21.02 (2021), p. 2150003. eprint: https://doi.org/10.1142/S0219061321500033. URL: https://doi.org/10.1142/S0219061321500033.

[She17] Saharon Shelah. *Proper and Improper Forcing*. 2nd ed. Perspectives in Logic. Cambridge University Press, 2017.

[She79] Saharon Shelah. "On Successors of Singular Cardinals". In: *Logic Colloquium '78*. Vol. 97. Elsevier, 1979, pp. 357–380. URL: https://www.sciencedirect.com/science/article/pii/S0049237X08716355.

[She91] Saharon Shelah. "Reflecting stationary sets and successors of singular cardinals". In: *Archive for Mathematical Logic* 31 (1991), pp. 25 –53.

[Ung12] Spencer Unger. "Fragility and Indestructibility of the Tree Property". In: *Archive for Mathematical Logic* 51.5-6 (2012), pp. 635–645.

[Ung15] Spencer Unger. "Fragility and indestructibility II". In: *Annals of Pure and Applied Logic* 166.11 (2015), pp. 1110–1122. URL: https://www.sciencedirect.com/science/article/pii/S0168007215000652.

[Ung17] Spencer Unger. "Successive failures of approachability". In: *Israel Journal of Mathematics* 242 (2017), pp. 663 –695. URL: https://api.semanticscholar.org/CorpusID:9808333.

[Via12] Matteo Viale. "Guessing models and generalized Laver diamond". In: *Annals of Pure and Applied Logic* 163.11 (2012). Kurt Goedel Research Prize Fellowships 2010, pp. 1660–1678. ISSN: 0168-0072. URL: https://www.sciencedirect.com/science/article/pii/S0168007211001886.

[VW11] Matteo Viale and Christoph Weiß. "On the consistency strength of the proper forcing axiom". In: *Advances in Mathematics* 228.5 (2011), pp. 2672–2687. ISSN: 0001-8708. URL: https://www.sciencedirect.com/science/article/pii/S0001870811002635.

[Wei10] Christoph Weiß. "Subtle and Ineffable Tree Properties". PhD thesis. 2010.

[Zem17] Martin Zeman. "Two Upper Bounds on Consistency Strength of $\neg\square_{\aleph_\omega}$ and Stationary Set Reflection at Two Successive $\aleph_n$". In: *Notre Dame Journal of Formal Logic* 58.3 (2017), pp. 409 –432. URL: https://doi.org/10.1215/00294527-2017-0005.