ITERATIONS OF (α, δ)-SEMIPROPER FORCINGS

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One of the incredible advancements made by Shelah is the concept of *countable support iterations*, which is a technique used to construct iterations of proper forcings while preserving the properness. One of the reasons countable support iterations work is that every proper forcing has the *countable covering property*, stating that every new countable set is covered by a countable set from the ground model. This ensures that the *factor property* holds, which states that a tail-segment of a countable support iteration of proper forcings is again a countable support iteration.

However, when forcing with a poset which is merely *semiproper*, this is no longer the case: For example, although it is semiproper, Prikry forcing adds a countable cofinal set to a former measurable cardinal κ without collapsing it, which clearly implies that this countable set cannot be covered by any $<\kappa$-sized set from the ground model. Due to this, countable support iterations of semiproper posets can fail to be semiproper and even collapse ω_1.

The remedy of this is, again, due to Shelah (see [She17, Chapter X]): When iterating semiproper posets, he instead uses *revised countable support iterations.* Roughly speaking, in such iterations, the support of a condition is no longer a countable set of ordinals, but a countable set of *names of ordinals.* As is to be expected, allowing names of ordinals to be used introduces other technical complications. Later on, a more manageable approach to the problem of iterating semiproper posets using Boolean algebras was found by Donder ([Fuc92]) and used e.g. by Jensen ([Jen]). A detailed write-up of this technique was produced by Viale-Audrito-Steila ([VAS14]).

In his book on proper and improper forcing, Shelah proves the iterability of α-properness with countable support and states that (α, ω_1)-semiproperness is iterable with revised countable support by the same proof ([She17, Theorem 1.8 and Theorem 1.9]). Since the technique of revised countable support iterations is still deemed very technical, we provide here a detailed write-up of his result using boolean algebraic iterations.

Definition 0.1. Let $\mathbb{P}$ be a poset and δ a cardinal. We say that $\mathbb{P}$ is *δ-semiproper* if for every large enough regular Θ, every countable $M \prec H(\Theta)$ with $\mathbb{P} \in M$ and every $p \in M \cap \mathbb{P}$ there is a condition $q \leq p$ which is $(M, \mathbb{P})$-δ-semigeneric, namely, $q \Vdash_{\mathbb{P}} M[\dot{G}] \cap \delta = M \cap \delta$.

The definition of α-properness, which α-semiproperness is modeled after, uses towers of models instead of single ones:

Definition 0.2. Let α be a countable ordinal and $\overline{M} = (M_i)_{i<\alpha}$ a sequence of countable elementary substructures of $H(\Theta)$, where Θ is regular uncountable. We say that $\overline{M}$ is an *α-tower* if

(1) For every limit $\delta < \alpha$, $M_\delta = \bigcup_{i<\delta} M_i$;
(2) For every successor $j < \alpha$,

$$(M_i)_{i<j} \in M_j$$

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Definition 0.3. Let $\mathbb{P}$ be a poset, α a countable ordinal and δ a cardinal. We say that $\mathbb{P}$ is α-semiproper if for every sufficiently large Θ, there is $x \in H(\Theta)$ such that for every α-tower $(M_i)_{i<\alpha}$ of countable elementary substructures of $H(\Theta)$, the following holds: Whenever $\mathbb{P}, x \in M_0$ and $p \in \mathbb{P} \cap M_0$, there is $q \leq p$ such that q is $(M_i, \mathbb{P})$-semigeneric for every $i < \alpha$.

Let us first note that α-semiproperness is preserved by two-step iterations:

Lemma 0.4. *Let α be a countable ordinal. Suppose that $\mathbb{P}$ is α-semiproper and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for an α-semiproper forcing. Then $\mathbb{P} * \dot{\mathbb{Q}}$ is α-semiproper.*

Proof. Let $(M_i)_{i<\alpha}$ be an α-tower of countable elementary submodels of $H(\Theta)$ where Θ is a sufficiently large regular cardinal such that $\mathbb{P} * \dot{\mathbb{Q}}, \alpha \in M_0$ and let $(p, \dot{q}) \in M_0 \cap \mathbb{P} * \dot{\mathbb{Q}}$. Let $p_0 \leq p$ be $(M_i, \mathbb{P})$–semigeneric for every $i < \alpha$. Let G be $\mathbb{P}$-generic containing p_0. Then $(M_i[G])_{i<\alpha}$ is an α-tower of countable elementary submodels of $H(\Theta)[G] = (H(\Theta))^{V[G]}$ and thus there is $\dot{q}_0^G \leq \dot{q}^G$ which is $M_i[G]$-semigeneric for every $i < \alpha$. Back in V, let $\dot{q}_0$ be forced to be an extension of $\dot{q}$ which is $M_i[\dot{G}_{\mathbb{P}}]$-semigeneric for every $i < \alpha$. We claim that $(p_0, \dot{q}_0)$ is M_i-semigeneric for every $i < \alpha$. Let $G * H$ be $\mathbb{P} * \dot{\mathbb{Q}}$-generic containing $(p_0, \dot{q}_0)$ and let $i < \alpha$. Since p_0 is $(M_i, \mathbb{P})$-semigeneric, $M_i[G] \cap \omega_1 = M_i \cap \omega_1$. Since $\dot{q}_0^G$ is $(M_i[G], \dot{\mathbb{Q}}^G)$-semigeneric, $M_i[G][H] \cap \omega_1 = M_i[G] \cap \omega_1$. In summary, we have

$$M_i[G * H] \cap \omega_1 = M_i[G][H] \cap \omega_1 = M_i[G] \cap \omega_1 = M_i \cap \omega_1$$

just as desired. ☐

Now we turn to longer iterations. To do so, we must first introduce some concepts related to the iteration of forcings using Boolean algebras.

The following is well-known:

Proposition 0.5. *Let $\mathbb{P}$ be a poset. Then there is a boolean algebra $\mathbb{B}(\mathbb{P})$ and a dense embedding $\iota\colon \mathbb{P} \to \mathbb{B}(\mathbb{P})$.*

Iterating with Boolean algebras requires us to move from the usual notion of iteration (where conditions are functions with a given support) to the following one:

Definition 0.6. Let $(\mathbb{B}_\alpha)_{\alpha<\gamma}$ be a sequence of complete Boolean algebras. We say that $(\mathbb{B}_\alpha)_{\alpha<\gamma}$ is an *iteration* if for each $\alpha < \beta < \gamma$, $\mathbb{B}_\alpha$ is a complete suborder of $\mathbb{B}_\beta$.

As we are working with Boolean algebras, the notions of regular embeddings and projections are dual to each other:

Definition 0.7. Let $(\mathbb{B}_\alpha)_{\alpha<\gamma}$ be an iteration. Let $\mathbb{B} := \bigcup_{\alpha<\gamma} \mathbb{B}_\alpha$ and let $h_\alpha\colon \mathbb{B} \to \mathbb{B}_\alpha$ be defined by

$$h_\alpha(b) := \bigwedge\{c \in \mathbb{B}_\alpha \mid b \leq c\}$$

We first record the following easy facts for later:

Lemma 0.8. *Let $(\mathbb{B}_\alpha)_{\alpha<\gamma}$ be an iteration. Let $b \in \bigcup_{\alpha<\gamma} \mathbb{B}_\alpha$ and let $\alpha < \gamma$.*

1. *if $b' \leq b$, then $h_\alpha(b') \leq h_\alpha(b)$.*
2. *$h_\alpha(b) = 0 \leftrightarrow b = 0$;*
3. *$h_\alpha\left(\bigvee_{i\in I} b_i\right) = \bigvee_{i\in I} h_\alpha(b_i)$;*
4. *for every $a \in \mathbb{B}_\alpha$, $h_\alpha(b \wedge a) = a \wedge h_\alpha(b)$.*

Proof. (1): If $b' \leq b$, then for every c with $b \leq c$, $b' \leq c$. Thus $h_\alpha(b')$ takes a conjunction of a larger set and is thus smaller.

(2): Clearly, if $b = 0$, then $b \leq c$ for all $c \in \mathbb{B}_\alpha$, which implies $h_\alpha(b) = 0$. On the other hand, let $b \neq 0$. Then there is $b' \in \mathbb{B}_\alpha$ such that every $c \leq b'$ in $\mathbb{B}_\alpha$ is compatible with b. We claim that $b' \leq h_\alpha(b)$. Otherwise, there is $c \in \mathbb{B}_\alpha$ with $b \leq c$ and $b' \not\leq c$. In particular, $b' \wedge \neg c \neq 0$. However, the latter condition is below b' and incompatible with c, a contradiction.

(3): By (1), for every $i \in I$, $h_\alpha(b_i) \leq h_\alpha\left(\bigvee_{i\in I} b_i\right)$ and thus $\bigvee_{i \in I} h_\alpha(b_i) \leq h_\alpha\left(\bigvee_{i\in I} b_i\right)$. On the other hand, let $c \in \mathbb{B}_\alpha$, $c \geq \bigvee_{i\in I} b_i$. Then $c \geq b_i$ for every $i \in I$ and thus $c \geq h_\alpha(b_i)$ for every $i \in I$, which implies $c \geq \bigvee_{i\in I} h_\alpha(b_i)$. Since c was arbitrary,

$$h_\alpha\left(\bigvee_{i\in I} b_i\right) = \bigwedge\left\{c \in \mathbb{B}_\alpha \,\middle|\, \bigvee_{i\in I} b_i \leq c\right\} \geq \bigvee_{i\in I} h_\alpha(b_i)$$

(4): We have the following:

$$\begin{aligned} h_\alpha(b \wedge a) &= \bigwedge\{c \in \mathbb{B}_\alpha \mid b \wedge a \leq c\} \\ &= \bigwedge\{c \wedge a \mid c \in \mathbb{B}_\alpha, b \leq c\} \\ &= \bigwedge\{c \in \mathbb{B}_\alpha \mid b \leq c\} \wedge a \\ &= h_\alpha(b) \wedge a \end{aligned}$$

For the second equality, notice that if $b \wedge a \leq c$, $b \leq c \vee \neg a$ and $(c \vee \neg a) \wedge a = c$, so every c with $b \wedge a \leq c$ can be written as $d \wedge a$ for $d \in \mathbb{B}_\alpha$ with $b \leq d$. ☐

Lemma 0.9. *Let* $(\mathbb{B}_\alpha)_{\alpha<\gamma}$ *be an iteration. Then for every* $\alpha < \beta < \gamma$, $h_\alpha = h_\alpha \circ h_\beta$.

Proof. Let $b \in \bigcup_{\alpha<\gamma} \mathbb{B}_\alpha$. By the definition,

$$h_\alpha(h_\beta(b)) = \bigwedge\{c \in \mathbb{B}_\alpha \mid h_\beta(b) \leq c\} = \bigwedge\left\{c \in \mathbb{B}_\alpha \,\middle|\, \bigwedge\{d \in \mathbb{B}_\beta \mid b \leq d\} \leq c\right\}$$

We prove that $h_\alpha(b) = h_\alpha(h_\beta(b))$.

On one hand, if $h_\beta(b) \leq c$, we have

$$b \leq \bigwedge\{d \in \mathbb{B}_\beta \mid b \leq d\} = h_\beta(b) \leq c$$

and so $b \leq c$. On the other hand, if $b \leq c$, then $c \in \{d \in \mathbb{B}_\beta \mid b \leq d\}$ and so

$$h_\beta(b) = \bigwedge\{d \in \mathbb{B}_\beta \mid b \leq d\} \leq c$$

Therefore,

$$\{c \in \mathbb{B}_\alpha \mid b \leq c\} = \{c \in \mathbb{B}_\alpha \mid h_\beta(b) \leq c\}$$

and thus both sets have the same infimum. ☐

Definition 0.10. Let $\overline{\mathbb{B}} = (\mathbb{B}_\alpha)_{\alpha<\gamma}$ be an iteration.

1. A *thread* in $\overline{\mathbb{B}}$ is a function $f \in \prod_{\alpha<\gamma} \mathbb{B}_\alpha$ such that for all $\alpha < \beta < \gamma$, $h_\alpha(f(\beta)) = f(\alpha)$. The collection of all threads is $\mathcal{T}(\overline{\mathbb{B}})$.
2. A thread f is *eventually constant* if there is $b \in \bigcup_{\alpha<\gamma} \mathbb{B}_\alpha$ such that for all $\alpha < \gamma$, $f(\alpha) = h_\alpha(b)$. The collection of all eventually constant threads is $\mathcal{C}(\overline{\mathbb{B}})$.
3. A thread f is *short* if it is either eventually constant or there is $\alpha < \gamma$ such that
$$f(\alpha) \Vdash_{\mathbb{B}_\alpha} \operatorname{cof}(\gamma) = \omega$$
The collection of all short threads is $\mathcal{S}(\overline{\mathbb{B}})$.

Definition 0.11. Let $\overline{\mathbb{B}} = (\mathbb{B}_\alpha)_{\alpha<\gamma}$ be an iteration and let $f, g \in \mathcal{T}(\overline{\mathbb{B}})$. We let $f \leq g$ if for all $\alpha < \gamma$, $f(\alpha) \leq_{\mathbb{B}_\alpha} g(\alpha)$.

Lemma 0.12. *Let $\overline{\mathbb{B}} = (\mathbb{B}_\alpha)_{\alpha<\gamma}$ be an iteration and let $\mathcal{C}(\overline{\mathbb{B}}) \subseteq F \subseteq \mathcal{T}(\overline{\mathbb{B}})$. Assume that*

$$\forall \alpha < \gamma \, \forall f \in F \, \forall b \in \mathbb{B}_\alpha (b \leq f(\alpha) \to \exists g \in F(g \leq f \wedge g \leq c(b)))$$

Then F is separative and the mappings $\mathbb{B}_\alpha \to F$, $b \mapsto c(b)$ are complete embeddings.

Proof. We first prove the separativity. To this end, let $f, g \in F$ and assume that $f \not\leq g$. In particular, there is $\alpha < \gamma$ such that $f(\alpha) \not\leq g(\alpha)$. Since $\mathbb{B}_\alpha$ is a Boolean algebra, it is separative and thus there is $b \in \mathbb{B}_\alpha$ such that $b \leq f(\alpha)$ and $b \perp_{\mathbb{B}_\alpha} g(\alpha)$. By assumption, there is $h \in F$ which extends $c(b)$ and f. In particular, $h(\alpha) = h_\alpha(b) = b$ and thus $h \perp g$ as required.

Now let $\alpha < \gamma$. Let $\iota_\alpha(b) := c(b)$. We prove that ι_α is a complete embedding. It is easy to see that ι_α preserves $\leq$. Let $\mathcal{A} \subseteq \mathbb{B}_\alpha$ be a maximal antichain. We prove that $\iota_\alpha[\mathcal{A}] \subseteq F$ is a maximal antichain. To this end, let $f \in F$. Since $\mathcal{A}$ is a maximal antichain in $\mathbb{B}_\alpha$, there is $c \in \mathcal{A}$ and $d \leq c, f(\alpha)$. By assumption, $\iota_\alpha(d)$ and f are compatible. Thus every element of F is compatible with some element of $\iota_\alpha[\mathcal{A}]$ which is what we wanted to show. □

Corollary 0.13. *Let $\overline{\mathbb{B}} = (\mathbb{B}_\alpha)_{\alpha<\gamma}$ be an iteration. Let F be either $\mathcal{T}(\overline{\mathbb{B}})$, $\mathcal{C}(\overline{\mathbb{B}})$ or $\mathcal{S}(\overline{\mathbb{B}})$. Then F is separative and the mappings $\mathbb{B}_\alpha \to F$, $b \mapsto c(b)$ are complete embeddings.*

Proof. We verify that F satisfies the hypothesis of Lemma 0.12.

Let $\alpha < \gamma$, $f \in F$ and $b \in \mathbb{B}_\alpha$ be such that $b \leq f(\alpha)$.

Define $g \in \prod_{\beta<\gamma} \mathbb{B}_\beta$ as follows:

$$g(\beta) := \begin{cases} h_\beta(b) & \beta \leq \alpha \\ f(\beta) \wedge b & \beta > \alpha \end{cases}$$

We first show:

Claim. $g \in \mathcal{T}(\overline{\mathbb{B}})$.

Proof. Let $\beta_0 < \beta_1 < \gamma$. If $\alpha < \beta_0$, we have

$$h_{\beta_0}(g(\beta_1)) = h_{\beta_0}(f(\beta_1) \wedge b) = h_{\beta_0}(f(\beta_1)) \wedge b = f(\beta_0) \wedge b = g(\beta_0)$$

If $\beta_1 < \alpha$, we have

$$h_{\beta_0}(g(\beta_1)) = h_{\beta_0}(h_{\beta_1}(b)) = h_{\beta_0}(b) = g(\beta_0)$$

Lastly, if $\beta_0 \leq \alpha < \beta_1$, we have

$$\begin{aligned} h_{\beta_0}(g(\beta_1)) &= h_{\beta_0}(h_\alpha(g(\beta_1))) \\ &= h_{\beta_0}(h_\alpha(f(\beta_1) \wedge b)) \\ &= h_{\beta_0}(h_\alpha(f(\beta_1)) \wedge b) \\ &= h_{\beta_0}(f(\alpha) \wedge b) \\ &= h_{\beta_0}(b) \\ &= g(\beta_0) \end{aligned}$$

We also show that $g(\beta) \neq 0$ for every $\beta < \gamma$. For $\beta \leq \alpha$, this follows from the assumption that $b \neq 0$. For $\beta > \alpha$, this follows from the fact that $h_\alpha(g(\beta)) = g(\alpha) \neq 0$. □

If $F = \mathcal{T}(\overline{\mathbb{B}})$, the previous claim means that we are done. If $F = \mathcal{C}(\overline{\mathbb{B}})$, there is $d \in \mathbb{B}$ such that $f = c(d)$. In this case, $g = c(d \wedge b)$ and thus in $\mathcal{C}(\overline{\mathbb{B}})$ as well. Lastly, if $F = \mathcal{S}(\overline{\mathbb{B}})$, then either $f \in \mathcal{C}(\overline{\mathbb{B}})$ (and we are done by the previous case) or there is $\beta < \gamma$ such that $f(\beta) \Vdash_{\mathbb{B}_\beta} \operatorname{cof}(\gamma) = \omega$. In this case, it is clear that $g(\beta)$ forces the same so that $g \in \mathcal{S}(\overline{\mathbb{B}})$. □

Definition 0.14. We define the following limits:

$$\begin{aligned}\mathrm{Inv}(\overline{\mathbb{B}}) &:= \mathbb{B}(\mathcal{T}(\overline{\mathbb{B}}))\\ \mathrm{Dir}(\overline{\mathbb{B}}) &:= \mathbb{B}(\mathcal{C}(\overline{\mathbb{B}}))\\ \mathrm{RCS}(\overline{\mathbb{B}}) &:= \mathbb{B}(\mathcal{S}(\overline{\mathbb{B}}))\end{aligned}$$

Definition 0.15. Let $(\mathbb{B}_i)_{i<\gamma}$ be an iteration. We say that $(\mathbb{B}_i)_{i<\gamma}$ is an RCS-*iteration* if for every limit $\delta < \gamma$, $\mathbb{B}_\delta = \mathrm{RCS}((\mathbb{B}_i)_{i<\delta})$.

The following is clear:

Theorem 0.16. *Let γ be an ordinal and let F be a function on γ. Then there is an* RCS-*iteration $(\mathbb{B}_i)_{i<\gamma+1}$ such that for every $i < \gamma$, $\mathbb{B}_{i+1} = \mathbb{B}_i * F(i)$, provided $F(i)$ is a $\mathbb{B}_i$-name for a boolean algebra.*

The reason behind working with complete Boolean algebras as opposed to simple partial orders is the *factor property*, which roughly states that tails of RCS-iterations are still RCS-iterations.

Definition 0.17. Let $\mathbb{B}$ and $\mathbb{C}$ be complete Boolean algebras such that $\mathbb{B}$ is a complete subalgebra of $\mathbb{C}$. Let G be $\mathbb{B}$-generic. We let $\mathbb{C}/G$ be the algebra of all equivalence classes of the form $[c]_G$, where $c \sim_G c'$ iff $c' \triangle c \in G$, with the usual operations.

Definition 0.18. Let $\overline{\mathbb{B}} = (\mathbb{B}_\alpha)_{\alpha<\gamma}$ be an iteration. Let $\alpha < \gamma$ and let G be $\mathbb{B}_\alpha$-generic. In $V[G]$, we define $\overline{\mathbb{B}}/G := (\mathbb{B}_\beta/G)_{\alpha<\beta<\gamma}$.

The following is clear:

Lemma 0.19. *Let $\overline{\mathbb{B}} = (\mathbb{B}_\alpha)_{\alpha<\gamma}$ be an iteration. Let $\alpha < \gamma$ and let G be $\mathbb{B}_\alpha$-generic. Then $\overline{\mathbb{B}}/G$ is an iteration.*

Lemma 0.20. *Let $\overline{\mathbb{B}} = (\mathbb{B}_\alpha)_{\alpha<\gamma}$ be an iteration. Let $\alpha < \gamma$ and let G be $\mathbb{B}_\alpha$-generic. Then*

$$\mathrm{RCS}(\overline{\mathbb{B}})/G \cong \mathrm{RCS}(\overline{\mathbb{B}}/G)$$

Proof. We will define a dense embedding from $\mathcal{S}(\overline{\mathbb{B}})/G$ into $\mathcal{S}(\overline{\mathbb{B}}/G)$. It then follows that there is a dense embedding from $\mathcal{S}(\overline{\mathbb{B}})/G$ into $\mathrm{RCS}(\overline{\mathbb{B}}/G)$. By the uniqueness of the Boolean completion, we thus obtain an isomorphism of $\mathrm{RCS}(\overline{\mathbb{B}})/G$ and $\mathrm{RCS}(\overline{\mathbb{B}}/G)$.

Let $[f] \in \mathcal{S}(\overline{\mathbb{B}})/G$. Let $\iota(f)$ be a function with domain (α, γ) such that $\iota(f)(\beta) = [f(\beta)]$. It follows that ι is well-defined.

We show that the image of ι is dense. To this end, let $\tau^G \in \mathcal{S}(\overline{\mathbb{B}}/G)$. By the definition of $\mathcal{S}(\overline{\mathbb{B}}/G)$, it follows that either $\tau^G \in \mathcal{C}(\overline{\mathbb{B}}/G)$ or there is $\beta \in (\alpha, \gamma)$ such that $\tau^G(\beta) \Vdash_{\mathbb{B}_\beta/G} \mathrm{cf}(\gamma) = \omega$. We check two cases:

Case A: $\tau^G \in \mathcal{C}(\overline{\mathbb{B}}/G)$. Let $[c]_G \in \mathbb{B}_\beta/G$ for $\beta < \gamma$ be such that $\tau^G = c([c]_G)$. Let f be a function on γ defined by $f(\beta) = [h_\beta(c)]_G = h_\beta([c]_G)$. Then $f \in \mathcal{C}(\overline{\mathbb{B}}/G)$ and $\iota([f]) = \tau^G$, so we are done.

Case B: $\tau^G \in \mathcal{T}(\overline{\mathbb{B}}/G)$. Therefore there is $\beta \in (\alpha, \gamma)$ such that $\tau^G(\beta) \Vdash_{\mathbb{B}_\beta/G} \mathrm{cf}(\gamma) = \omega$. Let $p \in G$ force that $\tau(\beta) \Vdash_{\mathbb{B}_\beta/\mathbb{B}_\alpha} \mathrm{cf}(\gamma) = \omega$. In V, let f be a function on γ defined as follows:

$$f(\delta) := \begin{cases} h_\delta(p) & \delta \leq \alpha \\ \bigvee_{c \in \mathbb{B}_\delta} p \wedge c \wedge [\![[c]_{\mathbb{B}_\alpha} = \tau(\check{\delta})]\!] & \delta > \alpha \end{cases}$$

We want to show that $f \in \mathcal{T}(\overline{\mathbb{B}})$. To this end, we first show that for every $\delta > \alpha$, $h_\alpha(f(\delta)) = f(\alpha) = p$. Clearly, $h_\alpha(f(\delta)) \leq p$. So assume that $p \not\leq h_\alpha(f(\delta))$. Then there is $r \leq p$ in $\mathbb{B}_\alpha$ which is incompatible with $h_\alpha(f(\delta))$. However, there must be $r' \leq r$ forcing $\tau(\check{\delta}) = [c]_{\mathbb{B}_\alpha}$ for some $c \in \mathbb{B}_\delta$ which means that $r' \wedge c$ is nonzero, $h_\alpha(r' \wedge c) = r' \wedge h_\alpha(c) \leq h_\alpha(f(\delta))$ and $h_\alpha(r' \wedge c) \leq r' \leq r$, a contradiction.

Now let $\alpha < \delta_0 < \delta_1 < \gamma$. We want to show that $h_{\delta_0}(f(\delta_1)) = f(\delta_0)$. We first show $h_{\delta_0}(f(\delta_1)) \leq f(\delta_0)$. To this end, we have that

$$h_{\delta_0}(f(\delta_1)) = \bigvee_{c \in \mathbb{B}_{\delta_1}} h_{\delta_0}(p \wedge c \wedge [[[c]_{\mathbb{B}_\alpha} = \tau(\check{\delta}_1)]]$$

However, since τ is forced to be in $\mathcal{T}(\overline{\mathbb{B}}/G)$, $[[[c]_{\mathbb{B}_\alpha} = \tau(\check{\delta}_1)]] \leq [[h_{\delta_0}([c]_{\mathbb{B}_\alpha}) = \tau(\check{\delta}_0)]]$. So, in particular,

$$\bigvee_{c \in \mathbb{B}_{\delta_1}} h_{\delta_0}(p \wedge c \wedge [[[c]_{\mathbb{B}_\alpha} = \tau(\check{\delta}_1)]] \leq \bigvee_{c \in \mathbb{B}_{\delta_1}} p \wedge h_{\delta_0}(c) \wedge [[[h_{\delta_0}(c)]_{\mathbb{B}_\alpha} = \tau(\check{\delta}_0)]] = f(\delta_0)$$

On the other hand, suppose that $h_{\delta_0}(f(\delta_1)) \not\geq f(\delta_0)$. In particular, there is $r \leq f(\delta_0)$ which is incompatible with $h_{\delta_0}(f(\delta_1))$. However, we can find $r' \leq h_\alpha(r')$ which decides the value of $\tau(\check{\delta}_1)$ to be equal to some $[c]_{\mathbb{B}_\alpha}$. It follows that r' also decides that $\tau(\check{\delta}_0) = [h_{\delta_0}(c)]_{\mathbb{B}_\alpha}$. However, this implies that $r' \wedge h_{\delta_0}(c)$ is nonzero (since r' and $h_{\delta_0}(c)$ must be compatible) and below r'. Furthermore, $r' \wedge c \leq f(\delta_1)$ and so $h_{\delta_0}(r' \wedge c) = r' \wedge h_{\delta_0}(c)$ witnesses that r and $h_{\delta_0}(f(\delta_1))$ are compatible, a contradiction.

Additionally, it follows that $f(\beta) \Vdash \operatorname{cf}(\gamma) = \omega$, so $f \in \mathcal{S}(\overline{\mathbb{B}})$. In $V[G]$, $[f]_G \in \mathcal{S}(\overline{\mathbb{B}})/G$ and $[f]_G \leq \tau$ by construction, so we are done.

□

We now show that suitable limits of α-semiproper posets of length $\leq \omega_1$ are once again α-semiproper. The assumption that the length of the limit is $\leq \omega_1$ is crucial at one step in the proof. This assumption is why, in any iteration of semiproper forcing, Levy-collapses are interleaved at every second step to ensure that all limits are either inaccessible or have length $\leq \omega_1$ (modulo reordering).

The proof is quite similar to the proof of the iterability of α-properness ([Abr10, Theorem 5.5])

Lemma 0.21. *Let $\overline{\mathbb{B}} = (\mathbb{B}_i)_{i<\gamma}$ be an iteration where $\gamma \leq \omega_1$ is a limit ordinal and let α be a countable ordinal such that for each $i < j < \gamma$, $\mathbb{B}_i$ forces that $\mathbb{B}_j/\mathbb{B}_i$ is α-semiproper. Suppose that for each countable $\delta < \gamma$, $\mathbb{B}_\delta = \operatorname{Inv}((\mathbb{B}_i)_{i<\delta})$. Then the following holds:*

1. *If $\gamma < \omega_1$, $\operatorname{Inv}(\overline{\mathbb{B}})$ is α-semiproper;*
2. *If $\gamma = \omega_1$, $\operatorname{Dir}(\overline{\mathbb{B}})$ is α-semiproper.*

Proof. We may assume for simplicity that $\mathbb{B}_0 = 2$ is the trivial Boolean algebra. We prove both statements simultaneously by induction on α. Let $\mathbb{B}_\gamma$ be either $\mathcal{T}(\overline{\mathbb{B}})$ or $\mathcal{C}(\overline{\mathbb{B}})$ depending on whether $\gamma < \omega_1$ or $\gamma = \omega_1$. Since $\mathbb{B}_\gamma$ is dense in the respective limit, it suffices to show that $\mathbb{B}_\gamma$ is α-semiproper.

Case A: $\alpha = 1$. So, morally speaking, an α-tower is simply a countable elementary substructure. Let $M \prec H(\Theta)$ be countable with $\mathbb{B}_\gamma \in M$ and let $p \in \mathcal{C}(\overline{\mathbb{B}}_\gamma) \cap M$. We will assume for simplicity that p is the trivial condition. Let $(\tau_n)_{n\in\omega}$ enumerate all $\mathbb{B}_\gamma$-names for countable ordinals lying in M and let $(\gamma_n)_{n\in\omega}$ be an increasing sequence of ordinals converging to $M \cap \gamma$ such that $\gamma_0 = 0$. By induction on $n < \omega$ we define $p_n \in \mathbb{B}_\gamma$ such that

1. $h_{\gamma_n}(p_n)$ is $(M, \mathbb{B}_{\gamma_n})$-semigeneric;
2. it is forced (over $\mathbb{B}_{\gamma_n}$) that $[p_n]_{\mathbb{B}_{\gamma_n}} \in M[\Gamma_{\mathbb{B}_{\gamma_n}}]$;
3. for all $k < n$, $p_n \leq p_k$;
4. for all $k < n$ and $\alpha \in (\gamma_k, \gamma_n]$, $h_\alpha(p_n) = h_\alpha(p_k)$;
5. for all $k < n$, $p_n \Vdash_{\mathbb{B}_\gamma} \tau_n < M[\Gamma_{\mathbb{B}_{\gamma_n}}] \cap \omega_1$.

Let p_0 be trivial. Since $\mathbb{B}_{\gamma_0}$ is trivial, p_0 is $(M,\mathbb{B}_{\gamma_n})$-semigeneric and it is clear that the other statements hold as well.

So suppose p_n has been defined. For $c\in\mathbb{B}_\gamma$, let $\varphi(c)$ state the following in the forcing language of $\mathbb{B}_{\gamma_n}$:

1. $h_{\gamma_n}(c)\leq h_{\gamma_n}(p_n)$;
2. $[c]_{\mathbb{B}_{\gamma_n}}$ decides $[\tau_n]_{\mathbb{B}_{\gamma_n}}$ to be an element of $M[\Gamma_{\mathbb{B}_{\gamma_n}}]$;
3. $h_{\gamma_{n+1}}([c]_{\mathbb{B}_{\gamma_n}})$ is $(M[\Gamma_{\mathbb{B}_{\gamma_n}}],\mathbb{B}_{\gamma_{n+1}}/\mathbb{B}_{\gamma_n})$-semigeneric;
4. $h_{\gamma_{n+1}}(c)$ forces that $[c]_{\mathbb{B}_{\gamma_{n+1}}}\in M[\Gamma_{\mathbb{B}_{\gamma_{n+1}}}]$

Claim 1. *For every $q\leq h_{\gamma_n}(p_n)$, there is c such that $h_{\gamma_n}(c)\leq q$ and $h_{\gamma_n}(c)\leq [[\varphi(c)]]$.*

Proof. Let $q\leq h_{\gamma_n}(p_n)$. Let G_n be $\mathbb{B}_{\gamma_n}$-generic containing q. By assumption, $[p_n]_{G_n}\in M[G_n]$. By elementarity, we can find $[a]_{G_n}\leq[p_n]_{G_n}$ in $M[G_n]$ which decides $[\tau_n]_{G_n}$ to be some element of $M[G_n]$. Since $[a]_{G_n}\in M[G_n]$, $[a]_{G_n}\in\mathbb{B}_\delta/G_n$ for some $\delta<M[G_n]\cap\omega_1$. By assumption, $M[G_n]\cap\omega_1=M\cap\omega_1$. Since $\mathbb{B}_{\gamma_{n+1}}/G_n$ is semiproper, there is a condition $[b]_{G_n}\leq h_{\gamma_{n+1}}([a]_{G_n})$ which is $(M[G_n],\mathbb{B}_{\gamma_{n+1}}/G_n)$-semigeneric.

Let $[d]_{G_n}:=[b]_{G_n}\wedge[a]_{G_n}$. Then $h_{\gamma_{n+1}}([d]_{G_n})=[b]_{G_n}$ and so $[d]_{G_n}$ is nonzero. In particular, d and q are compatible and so $c:=d\wedge q$ is nonzero.

We now have the following: $h_{\gamma_n}(c)\leq q\leq h_{\gamma_n}(p_n)$. $[c]_{G_n}$ decides $[\tau_n]_{G_n}$ to be some element of $M[G_n]\cap\omega_1$, since it extends $[a]_{G_n}$. $h_{\gamma_{n+1}}([c]_{G_n})$ is $(M[G_n],\mathbb{B}_{\gamma_{n+1}}/G_n)$-semigeneric, since it extends $h_{\gamma_{n+1}}([b]_{G_n})$ and lastly, $h_{\gamma_{n+1}}(c)$ forces that $[c]_{G_{n+1}}=[a]_{G_{n+1}}$ since we only modified a on $\mathbb{B}_{\gamma_{n+1}}$ and in particular, the equivalence class is in $M[\Gamma_{\mathbb{B}_{\gamma_{n+1}}}]$.

Thus, by strenthening c if required, we may assume that $h_{\gamma_n}(c)\leq[[\varphi(c)]]$. □

Now let p_{n+1} be defined as follows:

$$p_{n+1}(\alpha):=\begin{cases} h_{\gamma_n}(p_n) & \alpha\leq\gamma_n\\ \bigvee_{c\in\mathbb{B}_\gamma} h_{\gamma_k}(c)\wedge[[\varphi(c)]] & \alpha\in(\gamma_n,M\cap\omega_1)\end{cases}$$

We show that p_{n+1} is as required. First note that $p_{n+1}\in\mathbb{B}_{M\cap\omega_1}$.

Claim. *For every $\alpha\in(\gamma_n,M\cap\omega_1)$, $h_{\gamma_n}(p_{n+1}(\alpha))=h_{\gamma_n}(p_n)$.*

Proof. It is clear that $h_{\gamma_n}(p_{n+1}(\alpha))\leq h_{\gamma_n}(p_n)$ simply by the definition of $\varphi(c)$.

So suppose that $h_{\gamma_n}(p_n)\not\leq h_{\gamma_n}(p_{n+1}(\alpha))$ for some α. By the separativity of $\mathbb{B}_{\gamma_n}$, it follows that there is $q\leq h_{\gamma_n}(p_n)$ which is incompatible with $h_{\gamma_n}(p_{n+1}(\alpha))$. However, by the claim, there is c such that $h_{\gamma_n}(c)\leq q$ and $h_{\gamma_n}(c)\leq[[\varphi(c)]]$. In particular, $h_\alpha(c)\leq p_{n+1}(\alpha)$ and thus $h_{\gamma_n}(c)=h_{\gamma_n}(h_\alpha(c))$ is below both $h_{\gamma_n}(p_{n+1}(\alpha))$ and $h_{\gamma_n}(p_n)$, a contradiction. □

So $p_{n+1}\in\mathbb{B}_\gamma$. Clearly, $h_{\gamma_n}(p_n)$ forces that $[h_{\gamma_{n+1}}(p_{n+1})]_{\mathbb{B}_{\gamma_n}}$ is $(M[\Gamma_{\mathbb{B}_{\gamma_n}}],\mathbb{B}_{\gamma_{n+1}}/\mathbb{B}_{\gamma_n})$-semigeneric. Since $h_{\gamma_n}(p_n)$ is $(M,\mathbb{B}_{\gamma_n})$-semigeneric, it follows easily that $h_{\gamma_{n+1}}(p_{n+1})$ is $(M,\mathbb{B}_{\gamma_{n+1}})$-semigeneric. By the definition of $\varphi(c)$, it follows that it is forced that $[p_{n+1}]_{\mathbb{B}_{\gamma_{n+1}}}\in M[\Gamma_{\mathbb{B}_{\gamma_{n+1}}}]$. Also, p_{n+1} extends p_n by construction and for all $\alpha\leq\gamma_n$, $h_\alpha(p_{n+1})=h_\alpha(p_n)$ by the definition of p_{n+1}. Lastly, $[h_{\gamma_{n+1}}(p_{n+1})]_{\mathbb{B}_{\gamma_n}}$ is forced to decide τ_n to be an element of $M[\Gamma_{\mathbb{B}_{\gamma_n}}]$ by assumption, so we are done.

Now assume that we have defined $(p_n)_{n\in\omega}$. Let p_ω be defined as follows: For every $\alpha\in(\gamma_n,\gamma_{n+1})$, $p_\omega(\alpha)=h_\alpha(p_{n+1})$. It follows from the inductive hypothesis that $p_\omega\in\mathbb{B}_{M\cap\gamma}$ and thus that $p_\omega\in\mathbb{B}_\gamma$. Furthermore, clearly $p_\omega\leq p_n$ for every $n\in\omega$.

We are done after showing:

Claim. *p_ω is $(M,\mathbb{B}_\gamma)$-semigeneric.*

Proof. Let $\tau \in M$ be a $\mathbb{B}_\gamma$-name for a countable ordinal. Thus there is $n \in \omega$ such that $\tau = \tau_n$. By the inductive hypothesis, p_ω forces that τ_n is below $M[\Gamma_{\mathbb{B}_{\gamma_{n+1}}}] \cap \omega_1$. However, p_ω also extends p_{n+1} and $h_{\gamma_{n+1}}(p_{n+1})$ is $(M, \mathbb{B}_{\gamma_{n+1}})$-semigeneric and thus it forces that $M[\Gamma_{\mathbb{B}_{\gamma_{n+1}}}] \cap \omega_1 = M \cap \omega_1$. In summary, p_ω forces that τ_n is below $M \cap \omega_1$. As τ was arbitrary, we are done. $\square$

Case B: α is a limit ordinal. Let $(M_i)_{i<\alpha}$ be an α-tower such that $\mathbb{B}_\gamma \in M_0$ and let $p \in \mathbb{B}_\gamma \cap M_0$. We will assume for simplicity that p is the trivial condition. Fix an increasing sequence $(\alpha_n)_{n\in\omega}$ of successor ordinals converging to α. Also fix a sequence $(\gamma_n)_{n\in\omega}$ converging to $\sup(\sup(M_i \cap \omega_1)_{i<\alpha})$ such that $\gamma_0 = 0$ and for every $n \in \omega$, $\gamma_n \in M_{\alpha_n}$.

By induction on $n \in \omega$, we define $p_n \in \mathbb{B}_\gamma$ such that

1. p_n is $((M_i)_{i<\alpha_n}, \mathbb{B}_\gamma)$-semigeneric;
2. $h_{\gamma_n}(p_n)$ is $((M_i)_{i<\alpha}, \mathbb{B}_{\gamma_n})$-semigeneric;
3. $[p_n]_{\mathbb{B}_{\gamma_n}} \in M_{\alpha_n}[\Gamma_{\mathbb{B}_{\gamma_n}}]$;
4. for all $k < n$, $p_n \leq p_k$;
5. for all $k < n$ and $\beta \in (\gamma_k, \gamma_n]$, $h_\beta(p_n) = h_\beta(p_k)$;

To start, note that the trivial condition is $((M_i)_{i<\alpha}, \mathbb{B}_{\gamma_0})$-semigeneric and lies in M_0. By the inductive hypothesis applied in M_{α_0}, we can find $p_0 \in M_{\alpha_0}$ which is $((M_i)_{i<\alpha_0}, \mathbb{B}_\gamma)$-generic. It is clear that p_0 is as required.

Now suppose that p_n has been defined. For $c \in \mathbb{B}_\gamma$, let $\varphi(c)$ state the following in the forcing language of $\mathbb{B}_{\gamma_n}$:

1. $h_{\gamma_n}(c) \leq h_{\gamma_n}(p_n)$;
2. $[c]_{\mathbb{B}_{\gamma_n}}$ is $((M_i[\Gamma_{\mathbb{B}_{\gamma_n}}])_{\alpha_n \leq i < \alpha_{n+1}}, \mathbb{B}_\gamma)$-semigeneric;
3. $h_{\gamma_{n+1}}([c]_{\mathbb{B}_{\gamma_n}})$ is $((M_i[\Gamma_{\mathbb{B}_{\gamma_n}}])_{\alpha_{n+1} \leq i < \alpha}, \mathbb{B}_{\gamma_{n+1}}/\mathbb{B}_{\gamma_n})$-semigeneric;
4. $h_{\gamma_{n+1}}(c)$ forces that $[c]_{\mathbb{B}_{\gamma_{n+1}}} \in M[\Gamma_{\mathbb{B}_{\gamma_{n+1}}}]$.

Claim. *For every $q \leq h_{\gamma_n}(p_n)$, there is c such that $h_{\gamma_n}(c) \leq q$ and $h_{\gamma_n}(c) \leq [[\varphi(c)]]$.*

Proof. Let $q \leq h_{\gamma_n}(p_n)$. Let G_n be $\mathbb{B}_{\gamma_n}$-generic containing q. By assumption, $[p_n]_{G_n} \in M_{\alpha_n}[G_n]$. By the inductive assumption applied inside of $M_{\alpha_{n+1}}[G_n]$, we can find $[a]_{G_n} \leq [p_n]_{G_n}$ in $M_{\alpha_{n+1}}[G_n]$ which is $((M_i[G_n])_{\alpha_n \leq i < \alpha_{n+1}}, \mathbb{B}_\gamma)$-semigeneric. By the assumption that $\mathbb{B}_{\gamma_{n+1}}/G_n$ is semiproper, we can find $[b]_{G_n} \leq h_{\gamma_{n+1}}([a]_{G_n})$ which is $((M_i[G_n])_{\alpha_{n+1} \leq i < \alpha}, \mathbb{B}_{\gamma_{n+1}}/G_n)$-semigeneric.

Let $[d]_{G_n} := [b]_{G_n} \wedge [a]_{G_n}$. Then $h_{n+1}([d]_{G_n}) = [b]_{G_n}$ and so $[d]_{G_n}$ is nonzero. In particular, d and q are compatible and so $c := d \wedge q$ is nonzero.

It follows just as before that c is as required. $\square$

Now let p_{n+1} be defined as follows:

$$p_{n+1}(\alpha) := \begin{cases} h_\alpha(p_n) & \alpha \leq \gamma_n \\ \bigvee_{c \in \mathbb{B}_\gamma} h_{\gamma_k}(c) \wedge [[\varphi(c)]] & \alpha \in (\gamma_n, M \cap \omega_1) \end{cases}$$

We show that p_{n+1} is as required. Note that, just like before, we have $p_{n+1} \in \mathbb{B}_{M \cap \omega_1}$. The only thing left to show is the semigenericity.

To show that p_{n+1} is $((M_i)_{i<\alpha_{n+1}}, \mathbb{B}_\gamma)$-semigeneric, note that already p_n is $((M_i)_{i<\alpha_n}, \mathbb{B}_\gamma)$-semigeneric by the inductive hypothesis. Furthermore, $h_{\gamma_n}(p_n)$, which is $((M_i)_{i<\alpha}, \mathbb{B}_{\gamma_n})$-semigeneric, forces that $[p_{n+1}]_{\mathbb{B}_{\gamma_n}}$ is $((M_i[\Gamma_{\mathbb{B}_{\gamma_n}}])_{\alpha_n \leq i < \alpha_n}, \mathbb{B}_\gamma)$-semigeneric. Thus, p_{n+1} is $((M_i)_{\alpha_n \leq i < \alpha_{n+1}}, \mathbb{B}_\gamma)$-semigeneric as required.

To show that $h_{\gamma_{n+1}}(p_{n+1})$ is $((M_i)_{i<\alpha}, \mathbb{B}_{\gamma_{n+1}})$-semigeneric, we must only show that it is $((M_i)_{\alpha_{n+1} \leq i < \alpha}, \mathbb{B}_{\gamma_{n+1}})$-semigeneric. To see this, note that $h_{\gamma_n}(p_n)$ is $((M_i)_{\alpha_{n+1} \leq i < \alpha}, \mathbb{B}_{\gamma_n})$-semigeneric by the inductive assumption and it forces that $h_{\gamma_{n+1}}([p_{n+1}]_{\mathbb{B}_{\gamma_n}})$ is $((M_i[\Gamma_{\mathbb{B}_{\gamma_n}}])_{\alpha_{n+1} \leq i < \alpha}, \mathbb{B}_{\gamma_{n+1}}/\mathbb{B}_{\gamma_{n+1}})$-semigeneric, which clearly implies the desired statement.

Now let p_ω be defined as follows: For $\alpha \in [\gamma_n, \gamma_{n+1})$, $p_\omega(\alpha) = h_\alpha(p_{n+1})$. It follows that $p_\omega \in \mathbb{B}_\gamma$. Furthermore, for every $n \in \omega$, $p_\omega \leq p_n$ and so, p_ω is $(M_i, \mathbb{B}_\gamma)$-semigeneric for every $i < \alpha$ as required.

Case C: $\alpha = \alpha' + 1$, α' a successor. Let $(M_i)_{i<\alpha}$ be an α-tower such that $\mathbb{B}_\gamma \in M_0$ and let $p \in \mathbb{B}_\gamma \cap M_0$. By the inductive hypothesis, applied in $M_{\alpha'}$, we can find $p' \in M_{\alpha'}$ which is $((M_i)_{i<\alpha'}, \mathbb{B}_\gamma)$-semigeneric. By Case A, we can find $p'' \leq p'$ which is $(M_{\alpha'}, \mathbb{B}_\gamma)$-semigeneric. It follows easily that p'' is $((M_i)_{i<\alpha}, \mathbb{B}_\gamma)$-semigeneric.

Case D: $\alpha = \alpha' + 1$, α' a limit. Let $(M_i)_{i<\alpha}$ be an α-tower such that $\mathbb{B}_\gamma \in M_0$ and let $p \in \mathbb{B}_\gamma \cap M_0$. By the inductive hypothesis, let $p' \leq p$ be $((M_i)_{i<\alpha'}, \mathbb{B}_\gamma)$-semigeneric. Since $M_{\alpha'} = \bigcup_{i<\alpha'} M_i$, it follows easily that p' is $((M_i)_{i<\alpha}, \mathbb{B}_\gamma)$-semigeneric. □

The only possible limit left to consider is one of inaccessible length. By interleaving enough collapses, a theorem of Baumgartner allows us to assume that, in this situation, the forcing has a sufficiently nice chain condition.

Lemma 0.22. *Let $\overline{\mathbb{B}} = (\mathbb{B}_i)_{i<\gamma}$ be an iteration where γ is an inaccessible cardinal such that $|\mathbb{B}_i| < \gamma$ for every $i < \gamma$. Suppose that $\mathbb{B}_\gamma := \mathrm{Dir}(\overline{\mathbb{B}})$ has the γ-cc and that for every $i < \gamma$, $\mathbb{B}_i$ is α-semiproper. Then $\mathbb{B}_\gamma$ is α-semiproper.*

Proof. Let $(M_i)_{i<\alpha}$ be an α-tower such that $\mathbb{B}_\gamma \in M_0$ and let $p \in M_0 \cap \mathbb{B}_\gamma$. Let $\delta := \sup_{i<\alpha} \sup(M_i \cap \gamma)$. Then $\delta < \gamma$. Thus, by the inductive assumption, we can find a condition $p' \leq p$ which is $((M_i)_{i<\alpha}, \mathbb{B}_\delta)$-semigeneric. We will show that p' is $((M_i)_{i<\alpha}, \mathbb{B}_\gamma)$-semigeneric. To this end, let $i < \alpha$ and let $\tau \in M_i$ be a $\mathbb{B}_\gamma$-name for a countable ordinal. By elementarity, there is a maximal antichain $\mathcal{A}$ in M_i of conditions deciding τ. Since $\mathbb{B}_\gamma$ has the γ-cc and $\mathbb{B}_\gamma$ is the direct limit, we can apply elementarity to find an ordinal $\eta \in M_i$ such that $\mathcal{A} \subseteq \mathbb{B}_\eta$. In particular, $\eta < \delta$ and τ is equivalent to a $\mathbb{B}_\delta$-name in M_i. Thus, since p' is $(M_i, \mathbb{B}_\delta)$-generic, it forces that $\tau \in M_i \cap \omega_1$ as required. □

The following theorem summarizes this section. As before, note that we involve sufficiently many collapses in order to be able to assume that every limit we take has length $\leq \omega_1$.

Theorem 0.23. *Let $\overline{\mathbb{B}} = (\mathbb{B}_i)_{i<\gamma}$ be an iteration. Suppose the following:*

1. *For every $i < \gamma$,*
$$\Vdash_{\mathbb{B}_i} \mathbb{B}_{i+1}/\mathbb{B}_i \text{ is } \alpha\text{-semiproper}$$
2. *For every limit $\delta < \gamma$, $\mathbb{B}_\delta = \mathrm{RCS}((\mathbb{B}_i)_{i<\delta})$;*
3. *For every even ordinal $i < \gamma$, $\Vdash_{\mathbb{B}_{i+1}} |2^{\mathbb{B}_i}| \leq \omega_1$.*

Then for every $i < \gamma$, $\mathbb{B}_i$ is α-semiproper.

Proof. We will show by induction on $i < \gamma$ that for every $j < i$,
$$\Vdash_{\mathbb{B}_j} \mathbb{B}_i/\mathbb{B}_j \text{ is } \alpha\text{-semiproper}$$

Case A: $i = i' + 1$ is a successor. Let $j < i$. If $j = i'$, we are done by assumption. Otherwise, let G_j be $\mathbb{B}_j$-generic. It follows that $\mathbb{B}_i/G_j$ is forcing equivalent to $(\mathbb{B}_{i'}/G_j) * (\mathbb{B}_i/\mathbb{B}_{i'})$. By Lemma 0.4, $\mathbb{B}_i/G_j$ is α-semiproper.

Case B: i is either not inaccessible or there is $\delta < i$ such that $|\mathbb{B}_\delta| \geq i$. Let $j < i$ In this case, there must be a cardinal $\delta \in (j, i)$ such that $2^\delta \geq \mathrm{cf}(i)$: If there is $\delta < i$ such that $|\mathbb{B}_\delta| \geq i$, this is clear. Otherwise, i is either singular in which case $\delta = \mathrm{cf}(i)$ works or not a strong limit. It follows that $\Vdash_{\mathbb{B}_\delta} \mathrm{cf}(i) \leq \omega_1$. So let $((M_i)_{i<\alpha})$ be an α-tower and let $p \in M_0 \cap \mathbb{B}_i$.

Case B0: $h_\delta(p) \Vdash \mathrm{cf}(i) = \omega_1$. Let G_j be $\mathbb{B}_j$-generic containing $h_j(p)$ and let $[a]_{G_j} \leq h_\delta([h]_{G_j})$ be $((M_i[G_j])_{i<\alpha}, \mathbb{B}_\delta/G_j)$-semigeneric. Let G_δ be $\mathbb{B}_\delta/G_j$-generic containing $h_\delta(p)$. In $V[G_\delta]$, let $f\colon \omega_1 \to i$ be increasing with $f(0) \geq \delta$. It follows that $\mathbb{B}_i/G_\delta$ is forcing equivalent to $\operatorname{Dir}((\mathbb{B}_{f(\alpha)}/G_\delta)_{\alpha<\omega_1})$, since ω_1 is not collapsed. By the inductive assumption we have that for every $\alpha < \beta < \omega_1$,

$$\Vdash_{\mathbb{B}_{f(\alpha)}} \mathbb{B}_{f(\beta)}/\mathbb{B}_{f(\alpha)} \text{ is } \alpha\text{-semiproper}$$

By Lemma 0.21, $\mathbb{B}_i/G_\delta$ is α-semiproper and $[p]_{G_\delta} = [p]_{G_\delta} \in M[G_\delta]$, so we can find an $((M_k[G_\delta])_{k<\alpha}, \mathbb{B}_i/G_\delta)$-semigeneric condition $[q]_{G_\delta}$ extending $[p]_{G_\delta}$. In particular, q and a are compatible. It follows that $q \wedge a$ is $((M_k[G_j])_{k<\alpha}, \mathbb{B}_i/G_\delta)$-semigeneric.

Case B1: $h_\delta(p) \not\Vdash \operatorname{cf}(i) = \omega_1$. We may therefore assume that $h_\delta(p) \Vdash \operatorname{cf}(i) = \omega$. Let G_j be $\mathbb{B}_j$-generic containing $h_j(p)$ and let $[a]_{G_j} \leq h_\delta([h]_{G_j})$ be $((M_i[G_j])_{i<\alpha}, \mathbb{B}_\delta/G_j)$-semigeneric. Let G_δ be $\mathbb{B}_\delta$-generic containing $h_\delta(p)$. In $V[G_\delta]$, let $f\colon \omega \to i$ be increasing with $f(0) \geq \delta$. By the Factor Lemma 0.20, it follows that

$$\mathbb{B}_i/G_\delta = \operatorname{RCS}((\mathbb{B}_k)_{k<i})/G_\delta = \operatorname{RCS}((\mathbb{B}_k/G_\delta)_{\delta\leq k<i}) = \operatorname{Inv}((\mathbb{B}_{f(k)}/G_\delta)_{k\in\omega})$$

By the inductive assumption, we have that for every $k_0 < k_1 < \omega$,

$$\Vdash_{\mathbb{B}_{f(k_0)}/G_\delta} \mathbb{B}_{f(k_1)}/G_\delta \text{ is } \alpha\text{-semiproper}$$

Therefore, by Lemma 0.21, $\mathbb{B}_i/G_\delta$ is α-semiproper. Furthermore, $[a]_{G_\delta} = [p]_{G_\delta}$ and so we can find a $((M_k[G_\delta])_{k<\alpha}, \mathbb{B}_i/G_\delta)$-semigeneric condition extending $[a]_{G_\delta}$. As before, this allows us to find a $((M_k[G_j])_{k<\alpha}, \mathbb{B}_i/G_\delta)$-semigeneric condition extending $[p]_{G_j}$ as required.

Case C: i is inaccessible and $|\mathbb{B}_\delta| < i$ for every $\delta < i$. Let $j < i$ and let G_j be $\mathbb{B}_j$-generic. Clearly, there are stationarily many $\delta \in (j, i)$ such that $\operatorname{cf}(\delta) = \omega_1$. In particular, for any such δ,

$$\mathbb{B}_\delta/G_j = \operatorname{Dir}((\mathbb{B}_k/G_j)_{k<\delta})$$

It therefore follows from a theorem of Baumgartner that $\mathbb{B}_i/G_j$ has the i-cc. Therefore, by Lemma 0.22, $\mathbb{B}_i/G_j$ is α-semiproper. □

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