ON SHELAH'S APPROACHABILITY IDEAL

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Abstract. We solve a long-standing open problem of Shelah regarding the *Approachability Ideal* $I[\kappa^+]$. Let $\aleph_\gamma$ be a singular cardinal and $\mu \in (\mathrm{cf}(\gamma), \aleph_\gamma)$ a regular cardinal. Assuming appropriate large cardinal hypotheses, we construct a model of ZFC in which $\aleph_{\gamma+1} \cap \mathrm{cof}(\mu) \notin I[\aleph_{\gamma+1}]$ holds. This provides a definitive answer to a question of Shelah from the 1980's. Assuming large cardinals, we construct a model of ZFC in which Shelah's approachability property fails, simultaneously, at every singular cardinal. This is a major milestone in the solution of a question of Foreman and Magidor from the 80's.

1. Introduction

The study of singular cardinals stands out as one of the most fascinating areas of inquiry in modern set theory. It was Georg Cantor who first formalized the concept of the infinite and initiated the investigation of the continuum. As part of his pioneering analysis, Cantor formulated the *Continuum Hypothesis* (CH), which posits that the cardinality of $\mathbb{R}$ (i.e., $2^{\aleph_0}$) is as small as possible—namely, $\aleph_1$, the first uncountable cardinal. The status of CH within the broader mathematical paradigm remained elusive to many leading figures until Kurt Gödel [Göd40] showed that CH cannot be refuted from the standard axioms of mathematics (ZFC), and later Paul Cohen [Coh63, Coh64], inventing the set-theoretic method of *forcing*, showed that CH cannot be proved from ZFC either. These two results established the independence of CH from the standard foundation of mathematics, setting the tone for sixty years of groundbreaking discoveries in which the phenomenon of independence played a central role. With the subsequent development of forcing, Easton demonstrated that there are virtually no ZFC-provable restrictions on the behavior of the power-set function $\kappa \mapsto 2^\kappa$ [Eas70]. This inaugurated a series of results showing that, at least for (so-called) *regular cardinals*—such as $\aleph_0$ (the cardinality of the integers) or $\aleph_1$—ZFC alone settles little about cardinal arithmetic. Confounding all expectations, a series of groundbreaking results pioneered by Galvin–Hajnal [GH75], Silver [Sil75], Magidor [Mag77] and Shelah [She94] revealed that *singular cardinals* (such as $\aleph_\omega$ and $\aleph_{\omega_1}$) obey far deeper constraints, and that ZFC itself proves remarkable structural theorems. For instance, a celebrated result of Shelah shows that if $2^{\aleph_n} < \aleph_\omega$ holds for all integers n then there is a provable upper bound on $2^{\aleph_\omega}$. This is in stark contrast with power sets of regular cardinals, where no such upper bound can be established in ZFC alone as per the work of Easton [Eas70].

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The results of this paper were obtained in summer of 2025 and have ever since been presented in various conferences and seminars, including the Luminy Set Theory meeting, the Young Set Theory Workshop, the UNT Logic Seminar and the IIAS XXVI Midrasha School in Singular Cardinal Combinatorics held in Jerusalem.

Signs of the striking nature of singular cardinals have been found well beyond the realm of cardinal arithmetic. A paradigmatic example is Shelah's Singular Compactness Theorem [She75] in infinite abelian group theory, which shows that every abelian group G that is $\aleph_\omega$-generated and almost free must be free. Magidor–Shelah extensively studied such phenomena in their classical paper [MS94], and they are now considered part of the standard landscape of the field. Singular cardinals have also surfaced in functional analysis. Improving a classical, influential, work by Gowers and Maurey [GM93], Dodos, López-Abad, and Todorcevic [DLAT11] showed that, consistently, every Banach space of density character at most $\aleph_\omega$ contains an unconditional basic sequence—a result that is arguably optimal.

The present manuscript continues this major line of research in modern set theory by analyzing a related core construction: Shelah's approachability ideal, $I[\kappa]$.

Let κ be a regular uncountable cardinal. A set $S \subseteq \kappa$ is called *stationary* if it intersects every *club* set $C \subseteq \kappa$ – that is, it intersects every subset of κ that is closed and unbounded in its ordinal topology. Intuitively speaking, club sets correspond to *measure one* sets whereas stationary sets parallel sets having *positive measure*. Stationarity is a cornerstone concept in infinitary combinatorics which has been critically utilized in Module Theory [EM02, CP24], Cohomology [Ber21] or Analysis [Far11, Far19, MP17] among many other areas in mathematics. In set theory, the preservation of these objects after passing to generic extensions becomes pivotal in the proof of some central results in the field [Mag82, CFM01, PRS23, BHU24].

In his classical paper [She79], Shelah introduced (if implicitly) the *approachability ideal* $I[\kappa]$ and the corresponding notion of the *approachability property* AP_κ, asserting that $\kappa^+ \in I[\kappa^+]$ (i.e., $I[\kappa^+]$ is improper). Through an incisive analysis, Shelah bridges $I[\kappa]$ with the preservation of stationary sets under set-theoretic forcing. Specifically, he shows that if $S \subseteq \kappa \cap \operatorname{cof}(\lambda)$[1] is stationary and $S \in I[\kappa]$ then the stationarity of S is preserved by so-called $<\lambda^+$-closed posets (see Section 2). Thereby, under AP_κ any stationary set $S \subseteq \kappa^+ \cap \operatorname{cof}(\lambda)$ is preserved by $<\lambda^+$-closed forcing. This is somewhat sharp as if $S \notin I[\kappa]$, then even garden variety $<\lambda^+$-closed forcings, such as the *Levy collapse* $\operatorname{Coll}(\lambda, \kappa)$, destroy the stationarity of S.

Shelah showed that $I[\kappa]$ is a normal ideal and demonstrated that it is rich enough to contain *many* stationary sets. Namely, if $\lambda < \kappa$ are both regular and $\lambda^+ < \kappa$ then $I[\kappa]$ contains a stationary subset of $\kappa \cap \operatorname{cof}(\lambda)$. Moreover, if κ itself is a regular cardinal then $\kappa^+ \cap \operatorname{cof}(<\kappa) \in I[\kappa^+]$, and thus $I[\kappa^+]$ is completely determined by membership of stationary subsets of $\kappa^+ \cap \operatorname{cof}(\kappa)$. As usual, the situation with successors of singular cardinals is way more subtle – in this case ZFC can only establish $\kappa^+ \cap \operatorname{cof}(\leq \operatorname{cf}(\kappa)) \in I[\kappa^+]$. Whether or not this result is sharp constitutes a major problem and is one of the issues that is under investigation in this manuscript.

Ever since its conception, investigations of $I[\kappa]$ have constituted a major area of research in set theory. Today's understanding of this subject has been substantially enriched by the work of Shelah [She91, She93], Foreman–Magidor [FM95, FM97], Mitchell [Mit09], Krueger [Kru19], Sharon–Viale [SV10], Gitik–Sharon [GS08], Gitik–Rinot [GR12], Unger [Ung17], Cummings et al. [CFM+18], and Mohammadpour–Velickovic [MV21a], among many other authors. Current interest in $I[\kappa]$ extends beyond its original connection with the preservation of stationary sets. This is due

[1] $\kappa \cap \operatorname{cof}(\lambda)$ is the standard notation for the collection of ordinals $\alpha < \kappa$ with cofinality λ.

to the interplay between $I[\kappa]$ and other central principles in combinatorial set theory, such as Jensen's $\square_\kappa$-principle and its relatives, the Tree Property (TP) and Shelah's *PCF theory*.

The approachability ideal $I[\kappa]$ is intimately connected to two major questions in the field, which have served as an inspiration to the present work. Namely,

Question 1.1 (Shelah, 80's)**.** Let κ be a singular cardinal. Must the approachability ideal $I[\kappa^+]$ contain a club set relative to cofinality $\mathrm{cf}(\kappa)^{++}$?[2]

Question 1.2 (Foreman, Magidor, 80's)**.** Must there be a regular cardinal $\kappa \geq \aleph_2$ for which TP_κ fails?

In the first part of the manuscript we provide a definitive, negative, answer to Question 1.1 by proving the following consistency result:

Theorem A. *Assume the* GCH *holds and that there is a supercompact cardinal. Given a singular cardinal $\aleph_\gamma$ and a regular cardinal $\mu \in (\mathrm{cf}(\gamma), \aleph_\gamma)$ there is a model of* ZFC *where $\aleph_{\gamma+1} \cap \mathrm{cof}(\mu) \notin I[\aleph_{\gamma+1}]$ holds.*[3]

Assuming the consistency of certain *large cardinal axiom* called a *supercompact*, Shelah proved in [She79] the consistency of ZFC with $\aleph_{\omega+1} \cap \mathrm{cof}(\omega_1) \notin I[\aleph_{\omega+1}]$. Shelah's argument was limited to cofinality ω_1 (see page 9 for a discussion) and so the author asked whether or not the statement $\aleph_{\omega+1} \cap \mathrm{cof}(\omega_2) \notin I[\aleph_{\omega+1}]$ was consistent with ZFC [For05, Question 8.3]. This instance of Question 1.1 remained open for quite a long time until a breakthrough, made by the first author and Levine [JL25], provided a positive answer. Subsequently, the first author extended this result in [Jak25], constructing a model of ZFC in which, for every singular cardinal μ of countable cofinality and every regular cardinal $\gamma \in (\aleph_0, \mu)$, $\mu^+ \cap \mathrm{cof}(\gamma) \notin I[\mu^+]$. In spite of these recent major advances, Question 1.1 remained beyond the reach of the known methods. The reason is that the methods developed in [JL25, Jak25] were largely specific to singular cardinals with countable cofinality due to their dependence on *Namba-style forcings*, hence new and more sophisticated methods has to be developed. Among the novelties offered by this paper there is the construction of a poset $\mathbb{P}(\vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}})$ which plays a role analogue to the Namba-like forcing from [JL25, §4] which works for arbitrary singular cardinals. This poset is inspired by former constructions that stem from the theory of Prikry-type forcings. In the vernacular language of this field, $\mathbb{P}(\vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}})$ is an *ideal-based* version of a Magidor-product of *one-point Prikry forcings*. This poset has the crucial property that, in a model obtained via the two-step iteration $\mathbb{P}(\vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}}) * \dot{\mathrm{Coll}}(\mu, (\sup \vec{\kappa})^+)$, every cofinal function from μ into $(\sup \vec{\kappa})^+$ has an initial segment that is not covered by any ground-model set of size $< \sup \vec{\kappa}$. This is the key to show that in the corresponding generic extension the stationary set $(\sup \vec{\kappa})^+ \cap \mathrm{cof}(\mu)$ is non-approachable. Another technical novelty of the forcing $\mathbb{P}(\vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}})$ is that its definition hinges on ideals rather than ultrafilters. This makes the standard *Prikry analysis* of the poset ostensibly more subtle than usual.

[2]Shelah's original question was formulated for $\kappa = \aleph_\omega$ (see [She94, Problem 2.5, p.439] or [For05, Question 3.5]), as this was the first case known to be beyond the reach of the methods available at the time. The question posed here is the one that is highlighted by Shelah's body of work on $I[\kappa^+]$ and that it is not restricted to the case of $\kappa = \aleph_\omega$.

[3]In [She94, Problem 2.5, p.439], Shelah also asked whether there can be stationary sets, outside the approachability ideal, consisting of ordinals with inaccessible cofinality. Notice that our Theorem A also answers this question in the affirmative.

One of the key features of Prikry-type forcings is their amenability to a theory of iterations [Mag76, Git10]. After identifying $\mathbb{P}(\vec{\kappa},\vec{\mathcal{I}},\vec{\mathcal{B}})$ as a member of this family of posets we proceed to iterate the construction used in Theorem A to obtain a global result, which provides and even more general answer to Question 1.1:

Theorem B. *Assume the* GCH *and that there is a proper class of supercompact cardinals. Then there is a model of* ZFC *where* AP_κ *fails for every singular cardinal.*

Moreover, for each singular cardinal κ, there are unboundedly many regular cardinals $\delta<\kappa$ for which $\kappa^+\cap\operatorname{cof}(\delta)\notin I[\kappa^+]$.

Theorem B is tied to Foreman–Magidor's Question 1.2 via Jensen's *weak square principle* $\square^*_\kappa$. A classical result of Jensen [Jen72] states that $\square^*_\kappa$ is equivalent to the existence of a *special* κ^+-Aronszajn tree; that is, a κ^+-tree $(T,<_T)$ admitting a function $f\colon T\to\kappa$ such that $x<_T y$ implies $f(x)\neq f(y)$. The key feature of such trees is that they remain κ^+-Aronszajn (i.e., branchless) in any outer model of ZFC where κ^+ remains a cardinal. Consequently, special κ^+-Aronszajn trees provide strong failures of TP_{κ^+}. Since $\square^*_\kappa$ implies AP_κ, Theorem B yields the first model of ZFC in which no special κ^+-Aronszajn trees exist for any singular cardinal κ.

It is widely recognized among specialists that getting the Tree Property at the successor of every singular cardinal is one of the principal technical challenges in resolving Foreman–Magidor Question 1.2 (see e.g., [MS96, Nee09, Nee14, CHM+20a, CHM+20b]). The methods developed here thus represent a substantial advance toward the resolution of this second long-standing problem.

The paper is organized as follows. In Section 2 we present the reader with preliminary definitions and results concerning colorings on successors of singular cardinals and basic forcing facts for later use. In Section 3 we define the poset $\mathbb{P}(\vec{\kappa},\vec{\mathcal{I}},\vec{\mathcal{B}})$ and establish its key properties. After that, we prove Theorem A. As a bonus result, we use our methods to improve the main result of [GS08]. In Section 4 we construct the iteration that yields Theorem B. The paper is concluded with a section with open questions and concluding remarks. The manuscript is self-contained, and the notations herein follow the standard vernacular set theory. Only basic acquaintance with the theory of forcing and large cardinals is presumed.

2. Preliminaries

2.1. **Notations and conventions.** Our forcing convention is that $p\leq q$ means "p is stronger than q". Given cardinals $\kappa<\lambda$, $\mathrm{Coll}(\kappa,\lambda)$ (resp. $\mathrm{Coll}(\kappa,{<}\lambda)$) denotes Levy's forcing collapsing λ (resp. every cardinal in (κ,λ)) to κ, and $\mathrm{Add}(\kappa,\lambda)$ the Cohen forcing adjoinig λ-many subsets to κ. For regular cardinals $\lambda<\kappa$, we denote by E^κ_λ (or $\kappa\cap\operatorname{cof}(\lambda)$) the set of all ordinals $\alpha<\kappa$ with cofinality λ. For a regular cardinal Θ, $H(\Theta)$ stands for the collection of sets of hereditary cardinality less than Θ. Given a function f and a set A, $f[A]$ denotes $\{f(a)\mid a\in A\}$. If $\mathcal{I}$ is an ideal on a (non-empty) set X, we denote by $\mathcal{I}^+$ the collection of $\mathcal{I}$-*positive* sets; namely, $\mathcal{I}^+:=\{A\subseteq\mathcal{P}(X)\mid A\notin\mathcal{I}\}$. Finally, if κ is an uncountable regular cardinal, we say that $\mathcal{I}$ is $<\kappa$-complete if $\mathcal{I}$ is closed under $<\kappa$-sized unions of its members.

Some of the arguments in this paper will require the use of *large cardinals*. The standard reference in this matter is Kanamori's text [Kan94] where the reader is referred for a more comprehensive exposition. For the sake of completeness, we remind the definition of two key large cardinal notions: An uncountable cardinal κ is *measurable* if there is a non-pricipal κ-complete ultrafilter on κ. We say that

κ is *supercompact* if for each $\lambda \geq \kappa$ there is an elementary embedding $j\colon V \to M$ with *critical point* κ, $j(\kappa) > \lambda$ and M closed under λ-sequences of its members.

2.2. **Forcing.** In this section we provide a few preliminaries on the method of set-theoretic forcing. For the general theory of forcing we refer our readers to [Kun14].

Definition 2.1. Let $(\mathbb{P}, \leq_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}})$ be posets. A function $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a *projection* if the following holds:

(1) $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$.
(2) If $p' \leq_{\mathbb{P}} p$, then $\pi(p') \leq_{\mathbb{Q}} \pi(p)$.
(3) Whenever $q \leq_{\mathbb{Q}} \pi(p)$, there is $p' \leq_{\mathbb{P}} p$ such that $\pi(p') \leq_{\mathbb{Q}} q$.

If there exists a projection from $\mathbb{P}$ to $\mathbb{Q}$, any forcing extension by $\mathbb{Q}$ can be forcing extended to an extension by $\mathbb{P}$. This is made precise via the next definition:

Definition 2.2. Let $\mathbb{P}$ and $\mathbb{Q}$ be posets and $\pi\colon \mathbb{P} \to \mathbb{Q}$ a projection. Let G be $\mathbb{Q}$-generic. In $V[G]$, let $\mathbb{P}/G$ consist of all those elements $p \in \mathbb{P}$ with $\pi(p) \in G$, ordered as a suborder of $\mathbb{P}$. We let $\mathbb{P}/\mathbb{Q}$ be a $\mathbb{Q}$-name for $\mathbb{P}/\dot{G}$ and call $\mathbb{P}/\mathbb{Q}$ the *quotient forcing* of $\mathbb{P}$ by $\mathbb{Q}$.

Fact 2.3. *Let $\mathbb{P}$ and $\mathbb{Q}$ be poset and $\pi\colon \mathbb{P} \to \mathbb{Q}$ a projection. Let G be $\mathbb{Q}$-generic over V and let H be $\mathbb{P}/G$-generic over $V[G]$. Then, H is $\mathbb{P}$-generic over V and $G = \pi[H]$. In particular, $V[G][H] = V[H]$.* □

Projections onto iterations are often obtained by considering the so-called *termspace forcing* – an idea due to Laver and subsequently exploited by Foreman [For83b]:

Definition 2.4 (Laver)**.** Let $(\mathbb{P}, \leq)$ be a poset and let $(\dot{\mathbb{Q}}, \dot{\leq})$ be a $\mathbb{P}$-name for a poset. The poset $(\mathbb{T}(\mathbb{P}, \dot{\mathbb{Q}}), \prec)$ consists of all $\mathbb{P}$-names for elements of $\dot{\mathbb{Q}}$, ordered as follows: $\dot{q}' \prec \dot{q}$ if and only if $1_{\mathbb{P}} \Vdash \dot{q}' \dot{\leq} \dot{q}$.

Using standard arguments on names, one shows:

Lemma 2.5. *Let $\mathbb{P}$ be a poset and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a poset. The identity function is a projection from $\mathbb{P} \times \mathbb{T}(\mathbb{P}, \dot{\mathbb{Q}})$ onto $\mathbb{P} * \dot{\mathbb{Q}}$.* □

Recall that a poset $\mathbb{P}$ is *separative* if whenever $p_0 \not\leq p$, there is $p_1 \leq p$ such that p_0 and p_1 are incompatible. We also record the following easy lemma for later:

Lemma 2.6. *Let $\mathbb{P}$ be a poset and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a poset. If $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is separative, then $\mathbb{T}(\mathbb{P}, \dot{\mathbb{Q}})$ is separative.*

Proof. Let $\dot{q}_0, \dot{q} \in \mathbb{T}(\mathbb{P}, \dot{\mathbb{Q}})$ be such that $\dot{q}_0 \not\prec \dot{q}$, i.e. $1_{\mathbb{P}} \nVdash \dot{q}_0 \dot{\leq} \dot{q}$. Ergo there is $p \in \mathbb{P}$ such that $p \Vdash \dot{q} \not\dot{\leq} \dot{q}'$. Since $\dot{\mathbb{Q}}$ is forced to be separative, by the maximum principle we can find $\dot{q}'$ such that $p \Vdash (\dot{q}' \dot{\leq} \dot{q}) \wedge (\dot{q}' \perp \dot{q}_0)$. Now let $\dot{q}_1$ be such that $p \Vdash \dot{q}' = \dot{q}_1$ and conditions incompatible with p force $\dot{q}' = \dot{q}_1$. Then $\dot{q}_1 \prec \dot{q}$ and clearly there is no $\dot{q}''$ such that $1_{\mathbb{P}} \Vdash \dot{q}'' \dot{\leq} \dot{q}_0, \dot{q}_1$. In particular, $\dot{q}_0$ and $\dot{q}_1$ are $\prec$-incompatible. □

Definition 2.7. Let κ be an uncountable regular cardinal and $\mathbb{P}$ be a poset. We say that $\mathbb{P}$ is $<\kappa$*-closed* if every $\leq$-decreasing sequence $\langle p_\alpha \mid \alpha < \theta \rangle$ of conditions in $\mathbb{P}$, with $\theta < \kappa$, admits a $\leq$-lower bound. We say that $\mathbb{P}$ is $<\kappa$-directed closed if every directed subset $D \subseteq \mathbb{P}$ of size $<\kappa$ has a lower bound.[4]

[4]A set $D \subseteq \mathbb{P}$ is *directed* if any two conditions p, q are compatible via a member of D.

We will later use posets that do not have lower bounds for arbitrary sequences of a given length, but only those that were constructed according to a nice strategy.

Definition 2.8. Let $\mathbb{P}$ be a poset and γ an ordinal. The *completenss game* $\mathcal{G}(\mathbb{P}, \gamma)$ is defined as follows: The game lasts γ many rounds. COM starts by playing $1_{\mathbb{P}}$. At any stage δ, we have a position $(p_\alpha, q_\alpha)_{\alpha<\delta}$. Then COM has to play a condition p_δ which is a lower bound of $(q_\alpha)_{\alpha<\delta}$ and INC has to play a condition $q_\delta \leq p_\delta$. COM wins if they can continue to play for all γ many rounds. Otherwise, INC wins.

For a cardinal κ, we say that $\mathbb{P}$ is *κ-strategically closed* if COM has a winning strategy in $\mathcal{G}(\mathbb{P}, \kappa)$. We say that $\mathbb{P}$ is *$<\kappa$-strategically closed* if COM has a winning strategy in $\mathcal{G}(\mathbb{P}, \gamma)$ for every $\gamma < \kappa$.

Foreman was able to relate the distributivity of a poset $\mathbb{P}$ to the nonexistence of a winning strategy for INC in the completeness game. This characterization often proves very useful because it allows one to use distributivity assumptions to construct sequences $(p_\alpha)_{\alpha<\delta}$ where p_α depends on the history $(p_\beta)_{\beta<\alpha}$, knowing that this construction will at least succeed once:

Theorem 2.9 (Foreman [For83a])**.** *Suppose that $\mathbb{P}$ is a poset and κ is a cardinal. Then $\mathbb{P}$ is $<\kappa^+$-distributive*[5] *if and only if* INC *does not have a winning strategy in $\mathcal{G}(\mathbb{P}, \kappa+1)$.*

A critical fact connecting all the notions defined so far is due to Easton:

Fact 2.10 (Easton's Lemma)**.** *Let κ be an uncountable regular cardinal. Assume that $\mathbb{P}$ is κ-cc and $\mathbb{Q}$ is $<\kappa$-closed. Then:*

(1) $\Vdash_{\mathbb{P}\times\mathbb{Q}}$ "*κ is a regular uncountable cardinal*".
(2) $\Vdash_{\mathbb{P}}$ "$\check{\mathbb{Q}}$ *is $<\kappa$-distributive*".
(3) $\Vdash_{\mathbb{Q}}$ "$\check{\mathbb{P}}$ *is κ-cc*". □

Our posets will derive their regularity properties from being of *Prikry-type*.

Definition 2.11 (Gitik)**.** Let $(\mathbb{P}, \leq, \leq_0)$ be such that $(\mathbb{P}, \leq)$ and $(\mathbb{P}, \leq_0)$ are posets and $\leq$ refines $\leq_0$. We say that $(\mathbb{P}, \leq, \leq_0)$ *is a Prikry-type forcing* if whenever $p \in \mathbb{P}$ and σ is a sentence in the forcing language, there exists $q \leq_0 p$ which decides σ.

Equivalently, $(\mathbb{P}, \leq, \leq_0)$ is a Prikry-type forcing if and only if names for elements of the ordinal 2 can be decided using $\leq_0$-extensions (also called *pure* or *direct* extensions). This idea leads to the following definition:

Definition 2.12. Let $(\mathbb{P}, \leq, \leq_0)$ be a Prikry-type forcing and μ a cardinal.

(1) We say that $(\mathbb{P}, \leq, \leq_0)$ has *pure μ-decidability* if whenever τ is a $\mathbb{P}$-name and $p \in \mathbb{P}$ forces $\tau < \check{\mu}$, there is $\alpha < \mu$ and $q \leq_0 p$ such that $q \Vdash \tau = \check{\alpha}$.
(2) We say that $(\mathbb{P}, \leq, \leq_0)$ has *almost pure μ-decidability* if whenever τ is a $\mathbb{P}$-name and $p \in \mathbb{P}$ forces $\tau < \check{\mu}$, there is $\alpha < \mu$ and $q \leq_0 p$ such that $q \Vdash \tau < \check{\alpha}$.

We now turn to the concept of iterating Prikry-type forcings. The most important point here is that we want retain that any such iteration still has the Prikry property. A notion of iteration which satisfies this property was invented by Magidor in [Mag76]. These iterations can be defined with Easton support, non-stationary and full support, among others, but we will only use Easton-support iterations.

[5] I.e., for each $\theta < \alpha^+$ and dense open sets $\{D_\alpha \mid \alpha < \theta\}$ in $\mathbb{P}$, $\bigcap_{\alpha<\theta} D_\alpha$ is also dense.

Definition 2.13. Let $((\mathbb{P}_\alpha, \leq_\alpha, \leq_{\alpha,0}), (\dot{\mathbb{Q}}_\alpha, \dot{\leq}_\alpha, \dot{\leq}_{\alpha,0}))_{\alpha<\rho}$ be a sequence such that each $(\mathbb{P}_\alpha, \leq_\alpha, \leq_{\alpha,0})$ is a poset and each $(\dot{\mathbb{Q}}_\alpha, \dot{\leq}_\alpha, \dot{\leq}_{\alpha,0})$ is a $\mathbb{P}_\alpha$-name for a Prikry-type poset. We define the statement "$((\mathbb{P}_\alpha, \leq_\alpha, \leq_{\alpha,0}), (\dot{\mathbb{Q}}_\alpha, \dot{\leq}_\alpha, \dot{\leq}_{\alpha,0}))_{\alpha<\rho}$ is an Easton-support Magidor iteration of Prikry-type forcings of length ρ" by induction on ρ.

$((\mathbb{P}_\alpha, \leq_\alpha, \leq_{\alpha,0}), (\dot{\mathbb{Q}}_\alpha, \dot{\leq}_\alpha, \dot{\leq}_{\alpha,0}))_{\alpha<\rho}$ is an Easton-support Magidor iteration of Prikry-type forcings of length ρ if $((\mathbb{P}_\alpha, \leq_\alpha, \leq_{\alpha,0}), (\dot{\mathbb{Q}}_\alpha, \dot{\leq}_\alpha, \dot{\leq}_{\alpha,0}))_{\alpha<\rho'}$ is an Easton-support Magidor iteration of Prikry-type forcings of length ρ' for every $\rho' < \rho$ and moreover:

(1) If $\rho = \rho' + 1$, then $(\mathbb{P}_\rho, \leq_\rho) := (\mathbb{P}_{\rho'}, \leq_{\rho'}) * (\dot{\mathbb{Q}}_{\rho'}, \dot{\leq}_{\rho'})$ and $(p', \dot{q}') \leq_{\rho,0} (p, \dot{q})$ if and only if $p' \leq_{\rho',0} p$ and $p' \Vdash \dot{q}' \dot{\leq}_{\rho',0} \dot{q}$.
(2) If ρ is a limit, then $\mathbb{P}_\rho$ consists of all functions p on ρ such that
   (a) For all $\alpha < \rho$, $p \restriction \alpha \in \mathbb{P}_\alpha$,
   (b) If ρ is inaccessible and $|\mathbb{P}_\alpha| < \rho$ for every $\alpha < \rho$, then there is some $\beta < \rho$ such that for all $\gamma \in (\beta, \rho)$, $p \restriction \gamma \Vdash p(\gamma) = 1_{\dot{\mathbb{Q}}_\gamma}$.

   and the following holds:
   (i) $p' \leq_\rho p$ if and only if $p' \restriction \rho' \leq_{\rho'} p \restriction \rho'$ for every $\rho' < \rho$ and there exists a finite subset b such that whenever $\rho' \notin b$ and $p \restriction \rho' \nVdash p(\rho') = 1_{\dot{\mathbb{Q}}_{\rho'}}$, then $p' \restriction \rho' \Vdash p'(\rho') \dot{\leq}_{\rho',0} p(\rho')$.
   (ii) $p' \leq_{\rho,0} p$ if and only if $p' \leq_\rho p$ and the set b is empty.

We have the following (see [Git10, Section 6.3] or [JL25, Lemma 2.4]):

Lemma 2.14. *Let $((\mathbb{P}_\alpha, \leq_\alpha, \leq_{\alpha,0}), (\dot{\mathbb{Q}}_\alpha, \dot{\leq}_\alpha, \dot{\leq}_{\alpha,0}))_{\alpha<\rho}$ be an Easton support Magidor iteration of Prikry-type forcings of length ρ. Let μ be a cardinal.*

Assume that for each $\alpha < \rho$, $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ has (almost) pure $\check{\mu}$-decidability. Then for each $\alpha < \rho$, $\mathbb{P}_\alpha$ has (almost) pure $\check{\mu}$-decidability. □

Prikry-type forcings mostly derive their regularity properties from the Prikry property together with the closure of their pure extension ordering. In iterations, notice that the pure extension ordering on the iteration is not the same as the iteration of the pure extension orderings. Despite this fact, since the regular ordering $(\leq)$ refines the pure extension ordering $(\leq_0)$, one checks easily that the usual proof of the iterability of closure properties goes through:

Lemma 2.15. *Let μ be a cardinal. Let $((\mathbb{P}_\alpha, \leq_\alpha, \leq_{\alpha,0}), (\dot{\mathbb{Q}}_\alpha, \dot{\leq}_\alpha, \dot{\leq}_{\alpha,0}))_{\alpha<\rho}$ be an Easton support Magidor iteration of Prikry-type forcings of length ρ. If μ is below the first inaccessible cardinal and for each $\alpha < \rho$, $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is $<\check{\mu}$-directed closed, then each $\mathbb{P}_\alpha$ is $<\mu$-directed closed.* □

Lastly, we show the existence of a nice projection for Prikry-type forcings (similar to an idea of Foreman, see [For83b]).

Lemma 2.16. *Let $((\mathbb{P}_\alpha, \leq_\alpha, \leq_{\alpha,0}), (\dot{\mathbb{Q}}_\alpha, \dot{\leq}_\alpha, \dot{\leq}_{\alpha,0}))_{\alpha<\rho}$ be an Easton support Magidor iteration of Prikry-type forcings of length ρ. Let $\mathbb{T}$ denote the Easton support product $\prod_{\alpha<\kappa} \mathbb{T}((\mathbb{P}_\alpha, \leq_\alpha), (\dot{\mathbb{Q}}_\alpha, \dot{\leq}_{\alpha,0}))$. Then, the identity map establishes a projection from $\mathbb{T}$ onto the direct extension ordering on the Easton limit of the iteration $((\mathbb{P}_\alpha, \leq_\alpha, \leq_{\alpha,0}), (\dot{\mathbb{Q}}_\alpha, \dot{\leq}_\alpha, \dot{\leq}_{\alpha,0}))_{\alpha<\rho}$.*

Proof. Let $\mathbb{P}$ be the Easton-support Magidor limit of $((\mathbb{P}_\alpha, \leq_\alpha, \leq_{\alpha,0}), (\dot{\mathbb{Q}}_\alpha, \dot{\leq}_\alpha, \dot{\leq}_{\alpha,0}))$. Recall that $\mathbb{P}$ consists of all functions p on κ such that $p \restriction \alpha \in \mathbb{P}_\alpha$ for all $\alpha < \kappa$

and the support of p is an Easton set (i.e. it is bounded in every regular cardinal), directly ordered by $p \leq_0 q$ provided $p \restriction \alpha \leq_{\alpha,0} q \restriction \alpha$ for all $\alpha < \kappa$. Denote by $\leq_{\mathbb{T}}$ the ordering on $\mathbb{T}$.

It is clear that the identity is a function from $\mathbb{T}$ onto $\mathbb{P}$ and preserves $\leq_0$: Let $t_0, t_1 \in \mathbb{T}$ with $t_0 \leq_{\mathbb{T}} t_1$. Then t_0 is a function on κ such that for each $\alpha < \kappa$, $t_0(\alpha)$ is a $(\mathbb{P}_\alpha, \leq_\alpha)$-name such that $1_{\mathbb{P}_\alpha} \Vdash t_0(\alpha) \in \dot{\mathbb{Q}}_\alpha$. It follows by induction on α that $t_0 \restriction \alpha \in \mathbb{P}_\alpha$ for every $\alpha < \kappa$ since we use the same support in both cases. For any $\alpha < \kappa$, $1_{\mathbb{P}_\alpha} \Vdash t_0(\alpha) \leq_0 t_1(\alpha)$. Ergo $t_0 \restriction \alpha \Vdash_{(\mathbb{P}_\alpha, \leq_\alpha)} t_0(\alpha) \dot{\leq}_{\alpha,0} t_1(\alpha)$. It follows again by induction that $t_0 \restriction \alpha \leq_{\alpha,0} t_1 \restriction \alpha$ for every $\alpha < \kappa$.

Now we show that the identity is a projection (see Definition 2.1). First, $1_{\mathbb{T}} = 1_{\mathbb{P}}$, so Clause (1) is fulfilled. Let $p \in \mathbb{P}$ and $t \leq_0 p$. We want to find $p' \leq_{\mathbb{T}} p$ with $p' \leq_0 t$. By induction on α, let $p'(\alpha)$ be a $(\mathbb{P}_\alpha, \leq_\alpha)$-name for an element of $\dot{\mathbb{Q}}_\alpha$ forced by $p' \restriction \alpha$ to be equal to $t(\alpha)$ and by conditions incompatible with $p' \restriction \alpha$ to be equal to $p(\alpha)$. Then the support of p' is equal to the support of t and thus an Easton set. It follows by induction that $p' \leq_{\mathbb{T}} p$ and $p' \restriction \alpha \leq_{\alpha,0} t \restriction \alpha$ for all $\alpha < \kappa$: For any α, by induction, $p'(\alpha)$ is in any case forced (i.e., by $1_{\mathbb{P}_\alpha}$) to be below $p(\alpha)$: $p' \restriction \alpha$ is in particular $\leq$-below $t \restriction \alpha$ and thus forces $p'(\alpha) \leq_0 p(\alpha)$, and conditions incompatible with $p' \restriction \alpha$ outright force $p'(\alpha) = p(\alpha)$. It also follows that $p' \restriction \alpha \Vdash p'(\alpha) = t(\alpha)$ and thus $p' \restriction \alpha + 1 \leq_{\alpha+1,0} t \restriction \alpha + 1$. □

2.3. **Approachability.** In this section we collect some standard notations and facts related to *Shelah's approachability ideal*. For a thorough account of the matter, we refer to Eisworth's excellent handbook chapter [Eis10].

Definition 2.17. Let κ be a regular uncountable cardinal and $\bar{a} = \langle a_\alpha \mid \alpha < \kappa \rangle$ a sequence of bounded subsets of κ. A limit ordinal $\alpha < \kappa$ is called *approachable with respect to* $\bar{a}$ if there is an unbounded set $A \subseteq \alpha$ with $\mathrm{otp}(A) = \mathrm{cf}(\alpha)$ such that

$$\{A \cap \beta \mid \beta < \alpha\} \subseteq \{a_\beta \mid \beta < \alpha\}.$$

Definition 2.18 (Approachability ideal)**.** Let κ be a regular uncountable cardinal. A set $S \subseteq \kappa$ is in $I[\kappa]$ if there is a sequence $\bar{a} = \langle a_\alpha \mid \alpha < \kappa \rangle$ of bounded subsets of κ and a club $C \subseteq \kappa$ such that every $\alpha \in C \cap S$ is approachable with respect to $\bar{a}$.

The terminology *approachability ideal* is justified by the following fact:

Fact 2.19 (Shelah)**.** *$I[\kappa]$ is a normal ideal on κ.* □

It is not necessarily the case that $I[\kappa]$ is a proper ideal. In fact, in this paper we will construct a model where $I[\kappa^+]$ is improper for all singular cardinals κ. The assertion that $I[\kappa]$ is an improper ideal is known as the *Approachability Property*:

Definition 2.20 (Approachability)**.** Let κ be a (possibly singular) cardinal. The *approachability property* holds at κ if $\kappa^+ \in I[\kappa^+]$. We denote this by AP_κ.[6]

Other standard facts about the ideal $I[\kappa]$ and the approachability property that we may not use in the paper but collect for the reader's benefit are:

Fact 2.21 (Shelah)**.** *Let κ be a cardinal. Then, the following hold:*

(1) *For every two regular cardinals $\lambda < \kappa$ with $\lambda^+ < \kappa$ there is a stationary set $S \subseteq \kappa \cap \mathrm{cof}(\lambda)$ in $I[\kappa]$.* ([Cum05, Theorem 9.2])

[6] At a times, when we context does not lead to confusion, we will write AP to refer to the approachability property without having a specific cardinal in mind.

(2) $\kappa^+ \cap \mathrm{cof}(<\kappa) \in I[\kappa^+]$ *provided* κ *is regular.* ([She91, Lemma 4.4])
(3) $\kappa^+ \cap \mathrm{cof}(\leq \mathrm{cf}(\kappa)) \in I[\kappa^+]$ *provided* κ *is singular.* ([Eis10, Corollary 3.29])
(4) $\square^*_\kappa$ *implies* AP_κ. ([Cum05, p.262])
(5) *If* $S \subseteq \kappa \cap \mathrm{cof}(\mu)$ *is a stationary set,* $S \in I[\kappa]$ *and* $\mathbb{P}$ *is a* $<\mu^+$*-closed forcing then* $\Vdash_{\mathbb{P}}$ "$\check{S}$ *is stationary*". ([She79, Theorem 20]) □

If κ is regular then (2) implies that $I[\kappa^+]$ is completely determined by membership to it of stationary subsets of the *critical cofinality* (i.e., κ). In contrast, if κ is singular then there is a wide array of possible configurations for $I[\kappa^+]$, as only $\kappa^+ \cap \mathrm{cof}(\leq \mathrm{cf}(\kappa)) \in I[\kappa^+]$ is provable. In Theorem A we show that this is the case.

Even though the approachability ideal $I[\kappa]$ is always *large* (see (1) above), Shelah showed that the existence of supercompact cardinals imposes certain non-trivial restrictions. Namely,

Fact 2.22 (Shelah, [She79] and [Eis10, Theorem 3.20])**.** *Suppose that* κ *is a supercompact cardinal and* $\lambda > \kappa$ *is a singular cardinal with* $\mathrm{cf}(\lambda) < \kappa$*. Then, there is a singular cardinal* $\theta < \kappa$ *of cofinality* $\mathrm{cf}(\lambda)$ *and a stationary set* $S \subseteq \lambda^+ \cap \mathrm{cof}(\theta^+)$ *that is not in* $I[\lambda^+]$*. In particular,* AP_λ *fails.* □

Here we shall be preoccupied with the failure of AP_λ at singular cardinals λ. The standard way to produce the failure of the approachability property at a *down-to-earth* cardinal (such as $\aleph_\omega$) traces back to Shelah [She79, Conclusion 29]. The basic idea is to start with a supercompact cardinal κ, let $S \subseteq \kappa^{+\omega+1} \cap \mathrm{cof}(\theta^+)$ a stationary set exemplifying the failure of $\mathsf{AP}_{\kappa^{+\omega}}$ and then "bring it down" to $\aleph_\omega$. More concretely, by forcing with $\mathrm{Coll}(\aleph_0, \theta) * \mathrm{Coll}(\aleph_1, <\kappa)$ Shelah produced a generic extension where the original stationary set S remains stationary, non-approachable and $S \subseteq \aleph_{\omega+1} \cap \mathrm{cof}(\omega_1)$. However, the property $\aleph_{\omega+1} \cap \mathrm{cof}(\omega_1) \notin I[\aleph_{\omega+1}]$ was only known to be true in generic extensions where θ^+ was preserved as a cardinal. This posed problems regarding the possible cofinalities at which AP could fail at small cardinals, and also in iterating such constructions to achieve failures of AP at many cardinals simultaneously. Due to this, most research surrounding the failure of AP_λ for singular cardinals λ was focused on achieving this on a club of cardinals (see e.g. the works of Ben-Neria et al. [BNLHU20] and Gitik [Git21a]).

A standard way to force the failure of AP at small singular cardinals is based on a characterization of approachability via colorings, an idea that appeared in [She79].

Definition 2.23 (Colorings)**.** Let κ be singular and $d\colon [\kappa^+]^2 \to \mathrm{cf}(\kappa)$ a coloring.

(1) We say that d is *transitive* if for each $\alpha < \beta < \gamma < \kappa^+$,
$$d(\alpha, \gamma) \leq \max\{d(\alpha, \beta), d(\beta, \gamma)\}.$$
(2) We say that d is *normal* if for each $i < \mathrm{cf}(\kappa)$,
$$\sup_{\alpha < \kappa^+} |\{\beta < \alpha \mid d(\beta, \alpha) \leq i\}| < \kappa.$$
(3) An ordinal $\alpha \leq \kappa^+$ is *d-approachable* if there is a cofinal set $A \subseteq \alpha$ such that for each $\beta \in A$,
$$\sup\{d(\gamma, \beta) \mid \gamma \in A \cap \beta\} < \mathrm{cf}(\kappa).$$[7]

[7]Without loss of generality one may assume that $\mathrm{otp}(A) = \mathrm{cf}(\alpha)$.

Remark 2.24. Whenever $d\colon [\kappa^+]^2 \to \mathrm{cf}(\kappa)$ is a normal transitive coloring, κ^+ can never be d-approachable: Suppose that $A \subseteq \kappa^+$ is a cofinal set witnessing that κ^+ is d-approachable. Consider $f\colon A \to \mathrm{cf}(\kappa)$ the map $\alpha \mapsto \sup\{d(\beta,\alpha) \mid \beta \in A \cap \alpha\}$. Let $B \subseteq A$ of size κ^+ and $i < \mathrm{cf}(\kappa)$ a color such that $\mathrm{range}(f \restriction B) = \{i\}$. It follows that $\sup_{\alpha\in B} |\{\beta < \alpha \mid d(\beta,\alpha) \leq i\}| = \kappa^+$ thus contradicting the normality of d.

We have opted to included κ^+ in the definition of d-approachability as this ordinal can become d-approachable in a forcing extension where it is collapsed. This will legitimate assertions of the form "$p \Vdash_{\mathbb{P}} \check{\kappa}^+$ is d-approachable".

Example 2.25. To build intuition about the notion of d-approachability consider the following example. Let $\langle f_\alpha \mid \alpha < \kappa^+\rangle$ be a scale on a product $\prod_{i<\mathrm{cf}(\kappa)} \kappa_i$. Let

$$d(\alpha,\beta) := \min\{i < \mathrm{cf}(\kappa) \mid \forall j \geq i\, (f_\alpha(j) < f_\beta(j))\}.$$

This coloring is transitive (alas not normal). Given $\alpha < \kappa^+$ with $\mathrm{cf}(\alpha) > \mathrm{cf}(\kappa)$, it follows that α is d-approachable if and only if α is a good point of $\langle f_\alpha \mid \alpha < \kappa^+\rangle$.

Fact 2.26 (cf. [Eis10, §3])**.** *Normal transitive colorings always exist.* □

The following characterization appears in [Eis10, Corollary 3.27]:

Fact 2.27 (Shelah)**.** *An ordinal $\alpha < \kappa^+$ is d-approachable if either $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\kappa)$ or there is an unbounded set $A \subseteq \alpha$ and a color $i < \mathrm{cf}(\kappa)$ such that for each $\beta < \gamma$ in A, $d(\beta,\gamma) \leq i$.* □

Additionally we will use the following fact due to Shelah, which vastly simplifies proofs of the preservation of non-approachability.

Fact 2.28 ([She79, Remark 28])**.** *Suppose $\alpha < \kappa^+$ is d-approachable. Whenever $A \subseteq \alpha$ is unbounded, there is $B \subseteq A$ unbounded such that $d \restriction [B]^2$ is bounded.* □

Given a transitive normal coloring $d\colon [\kappa^+]^2 \to \mathrm{cf}(\kappa)$, we denote

$$S(d) := \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) > \mathrm{cf}(\kappa) \wedge \exists A \subseteq \alpha \,\text{unbounded}\, \exists i < \mathrm{cf}(\kappa)\, d \restriction [A]^2 \subseteq i+1\}.$$

Fact 2.29 (Shelah, [Eis10, Corollary 3.35])**.** *Suppose that κ is a strong limit singular cardinal and $d\colon [\kappa^+]^2 \to \mathrm{cf}(\kappa)$ is a normal transitive coloring. Then,*

$$S(d) \cup (\kappa^+ \cap \mathrm{cof}(\mathrm{cf}(\leq\kappa)))$$

generates $I[\kappa^+]$ modulo NS_{κ^+}. □

The above shows that the property of a set $S \subseteq \kappa^+$ being in $I[\kappa^+]$ is independent of the coloring d chosen, for any two such colorings d_0 and d_1, the corresponding sets $S(d_0)$ and $S(d_1)$ are equal modulo a club subset of κ^+.

3. On Shelah's problem

In [JL25] the first author and Levine answered [She94, Problem 2.5, p.439] by producing a model of ZFC where $\aleph_{\omega+1} \cap \mathrm{cof}(\omega_2) \notin I[\aleph_{\omega+1}]$. Moreover, for each regular cardinal $\mu \in (\omega_1, \aleph_\omega)$, their construction gives a generic extension where $\aleph_{\omega+1} \cap \mathrm{cof}(\mu) \notin I[\aleph_{\omega+1}]$. In this section we present a forcing construction that solves Shelah's Question 1.1 in full generality: Given a singular cardinal $\aleph_\gamma$ and a regular cardinal $\mu \in (\mathrm{cf}(\gamma), \aleph_\gamma)$, we produce a model where $\aleph_{\gamma+1} \cap \mathrm{cof}(\mu) \notin I[\aleph_{\gamma+1}]$.

The forcing notion will be a two-step iteration consisting of a Magidor-style product $\mathbb{P}(\vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}})$ followed by a Levy collapse $\mathrm{Coll}(\mu, (\sup\vec{\kappa})^+)$. The poset $\mathbb{P}(\vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}})$ is an ideal-based version of a Magidor product of *one-point Prikry forcings*. To

guarantee that this poset satisfies both the Prikry and strong Prikry properties and works as expected we will impose certain restrictions on the input parameters $(\vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}})$. These restrictions are captured by an enhancement of the *Laver Ideal Property* (LIP) from [JL25] which, in abuse of notation, we keep denoting by LIP.

Definition 3.1. Let $\mu < \nu$ be regular cardinals. We let $\mathsf{LIP}(\mu, \nu)$ state that there exists $\mathcal{I}$, a $<\nu$-complete and normal ideal over ν such that there is a set $\mathcal{B} \subseteq \mathcal{I}^+$ which is dense in $\mathcal{I}^+$ with respect to $\subseteq$ and $(\mathcal{B}, \subseteq)$ is $<\mu$-directed closed.[8]

A list of properties that suffice for a poset to force instances of LIP are:

Definition 3.2. Let $\mu < \nu$ be regular cardinals with ν measurable. Let $\mathbb{P}$ be a poset. We say that $\mathbb{P}$ is a $\mathsf{LIP}(\mu, \nu)$*-forcing* if the following hold:

(1) $\mathbb{P}$ is separative, $<\mu$-directed closed, ν-cc and is a subset of V_ν.
(2) There is a club $C \subseteq \nu$ such that whenever $\alpha \in C$ is inaccessible, there is a projection $\pi \colon \mathbb{P} \to (\mathbb{P} \cap V_\alpha)$ such that whenever $X \subseteq \mathbb{P}$ is directed, $|X| < \mu$ and $p \in \mathbb{P} \cap V_\alpha$ satisfies $p \leq \pi(q)$ for every $q \in X$, there is $p^* \in \mathbb{P}$ such that $\pi(p^*) = p$ and $p \leq q$ for every $q \in X$.

Remark 3.3. $\mathrm{Coll}(\mu, <\nu)$ is a $\mathsf{LIP}(\mu, \nu)$-forcing whenever ν is measurable.

The proof of the following is as in [GJM78] and was first noticed by Laver:

Lemma 3.4 ([Jak25, Lemma 3.3])**.** *Let $\mu < \nu$ be regular cardinals such that ν is measurable. Let $\mathbb{P}$ be a $\mathsf{LIP}(\mu, \nu)$-forcing. Then $\mathbb{P}$ forces $\mathsf{LIP}(\mu, \nu)$.* □

For later use we prove the following easy lemma:

Lemma 3.5. *Let $\mu < \nu$ be regular cardinals such that ν is measurable. Suppose that $\mathbb{P}$ is a $\mathsf{LIP}(\mu, \nu)$-forcing and that $\mathbb{Q}$ is a $<\mu$-directed-closed forcing with $|\mathrm{TC}(\mathbb{Q})| < \nu$. Then, $\mathbb{P} \times \mathbb{Q}$ is a $\mathsf{LIP}(\mu, \nu)$-forcing.*

Proof. We go over the clauses of being a $\mathsf{LIP}(\mu, \nu)$-forcing. First, $\mathbb{P} \times \mathbb{Q}$ is a separative, $<\mu$-directed-closed, ν-cc poset contained in V_ν. Let $C \subseteq \nu$ be a club witnessing that $\mathbb{P}$ is a $\mathsf{LIP}(\mu, \nu)$-forcing. Put $C^* := C \setminus |\mathrm{TC}(\mathbb{Q})|^+$ and for each inaccessible $\alpha \in C^*$ define $\pi^* \colon (\mathbb{P} \times \mathbb{Q}) \to (\mathbb{P} \times \mathbb{Q}) \cap V_\alpha$ by $(p, q) \mapsto (\pi(p), q)$ where π is the projection map obtained for the $\mathsf{LIP}(\mu, \nu)$ property of $\mathbb{P}$. It is routine to check that π^* is a projection with the desired property. □

We will later consider an iteration of LIP-forcings. For technical reasons, we will need to show that the corresponding termspace forcings (Definition 2.4) also force LIP. This is provided by the following lemma:

Lemma 3.6. *Let $\mathbb{P}$ be a poset and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a poset. Let $\mu < \nu$ be regular cardinals such that ν is measurable. Assume that $|\mathbb{P}| < \nu$ and $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is a $\mathsf{LIP}(\check{\mu}, \check{\nu})$-forcing. Then $\mathbb{T}(\mathbb{P}, \dot{\mathbb{Q}})$ is a $\mathsf{LIP}(\mu, \nu)$-forcing.*

Proof. First of all, $\mathbb{T}(\mathbb{P}, \dot{\mathbb{Q}})$ is $<\mu$-directed closed by the *Maximum Principle* and separative by Lemma 2.6. The small size of $\mathbb{P}$ easily implies that $\mathbb{T}(\mathbb{P}, \dot{\mathbb{Q}}) \subseteq V_\nu$. Additionally, in combination with the measurability of ν, it also implies that $\mathbb{T}(\mathbb{P}, \dot{\mathbb{Q}})$ is ν-cc: Assume that $\mathcal{A} \subseteq \mathbb{T}(\mathbb{P}, \dot{\mathbb{Q}})$ is an antichain with size ν. For $\dot{q}, \dot{q}' \in \mathcal{A}$, there exists $p \in \mathbb{P}$ such that $p \Vdash \dot{q} \perp \dot{q}'$. (Otherwise, we would have $1_\mathbb{P} \Vdash \dot{q} || \dot{q}'$ and could

[8] The difference compared to the version of LIP considered in [JL25, Definition 3.1] is that here we require B to be $<\mu$-**directed** closed.

apply the maximum principle to find a condition witnessing the compatibility.) So we have a map from $[\mathcal{A}]^2 \to \mathbb{P}$. Since $|\mathcal{A}| = \nu$, which is measurable, and $|\mathbb{P}| < \nu$, there is $\mathcal{B} \subseteq \mathcal{A}$ with size ν and $p \in \mathbb{P}$ such that for every $\dot{q}, \dot{q}' \in \mathcal{B}$, $p \Vdash \dot{q} \perp \dot{q}'$. But then p forces that $\mathcal{B}$ is an antichain in $\dot{\mathbb{Q}}$, a contradiction.

So we have to find the projection. Again, the small size of $\mathbb{P}$ implies that for almost all α, $\mathbb{T}(\mathbb{P}, \dot{\mathbb{Q}} \cap \dot{V}_\alpha) = (\mathbb{T}(\mathbb{P}, \dot{\mathbb{Q}})) \cap V_\alpha$. So if we let $\dot{\pi}_\alpha$ be a $\mathbb{P}$-name for the projection $\dot{\mathbb{Q}} \to \dot{\mathbb{Q}} \cap \dot{V}_\alpha$, we can map $\dot{q}$ to $\dot{\pi}_\alpha(\dot{q}_\alpha)$ and obtain a projection from $\mathbb{T}(\mathbb{P}, \dot{\mathbb{Q}})$ to $\mathbb{T}(\mathbb{P}, \dot{\mathbb{Q}}) \cap V_\alpha$. (Here we again use the *Maximum Principle*.) Let $X \subseteq \mathbb{T}(\mathbb{P}, \dot{\mathbb{Q}})$ be directed, $|X| < \mu$ and $\dot{q} \in \mathbb{T}(\mathbb{P}, \dot{\mathbb{Q}}) \cap V_\alpha$ with $\dot{q} \prec \dot{\pi}_\alpha(\dot{r})$ for every $\dot{r} \in X$. It follows that $1_\mathbb{P}$ forces that $\check{X}$ is directed and $\dot{q} \check{\leq} \dot{\pi}_\alpha(\dot{r})$ for every $\dot{r}$ in X. By the maximum principle we can fix a $\mathbb{P}$-name $\dot{q}^*$ such that $1_\mathbb{P}$ forces $\dot{\pi}_\alpha(\dot{q}^*) = \dot{q}$ and $\dot{q}^* \leq \dot{r}$ for every $\dot{r} \in \check{X}$. Then $\dot{q}^*$ is easily seen to be as required. □

We now define the following enhancement of LIP which is necessary for the definition of the main Prikry-type forcing of this section:

Definition 3.7. An increasing sequence of regular cardinals $\langle \mu \rangle^\frown \langle \kappa_i \mid i < \gamma \rangle$ with $\gamma < \mu$ limit witnesses the *Simultaneous Laver Ideal Property*, $\mathrm{SLIP}(\mu, \langle \kappa_i \mid i < \gamma \rangle)$, if the following properties hold true for each ordinal $i < \gamma$:

(1) $\langle \kappa_i \mid i < \gamma \rangle$ is discrete; to wit, $2^{\sup_{j<i} \kappa_j} < \kappa_i$ holds for all $i < \gamma$.
(2) There is a $<\kappa_i$-complete ideal $\mathcal{I}_i$ over κ_i and a dense set $\mathcal{B}_i \subseteq \mathcal{I}_i^+$ with respect to $\subseteq$ such that the poset $(\mathcal{B}_i, \subseteq)$ is $<\mu$-directed closed.
(3) For every $i < \gamma$, $(\prod_{j \in [i,\gamma)} \mathcal{B}_j, \subseteq)$ is $<(2^{\sup_{\ell<i} \kappa_\ell})^+$-distributive.[9]

Remark 3.8. We emphasize that γ is not assumed to be regular, but just a limit ordinal. This will be useful later to get instances of failure of the approachability property at successors of singulars of the form $\aleph_{\omega+\omega}$ or $\aleph_{\omega_1+\omega}$.

The previous property is a weakening of the requirement that $\mathsf{LIP}(\kappa_i^+, \kappa_{i+1})$ holds for every ordinal $i < \gamma$. The point is that in the construction of the forcing iteration leading to Theorem A we will only be able to establish that the weaker property $\mathrm{SLIP}(\mu, \langle \kappa_i \mid i < \gamma \rangle)$ holds. As a result, we have to carry out our proofs using this weaker assumption. The basic reason behind this is that, for an increasing sequence $\langle \kappa_i \mid i < \gamma \rangle$ of measurable cardinals and $\mu < \kappa_0$ regular, the iteration of Levy-collapses collapsing κ_0 to μ and everything between $(\sup_{j<i} \kappa_j)^+$ and κ_i to $(\sup_{j<i} \kappa_j)^+$ forces $\mathsf{LIP}((\sup_{j<i} \kappa_j)^+, \kappa_i)$; in constrast, the product of the corresponding Levy-collapses only forces $\mathrm{SLIP}(\mu, \langle \kappa_i \mid i < \gamma \rangle)$:

Lemma 3.9. *Let γ be a limit ordinal and let $\langle \kappa_i \mid i < \gamma \rangle$ be an increasing sequence of measurable cardinals. Let $\mu \in (\gamma, \kappa_0)$ be regular. For each $i < \gamma$, let $\mathbb{P}_i$ be a $\mathsf{LIP}((2^{\sup_{j<i} \kappa_j})^+, \kappa_i)$-forcing. Then the Easton support product $\mathbb{Q} := \prod_{i<\gamma} \mathbb{P}_i$ forces $\mathrm{SLIP}(\mu, \langle \kappa_i \mid i < \gamma \rangle)$.*

Proof. For each $\zeta < \gamma$, let $\mathbb{Q}_\zeta := \prod_{i<\zeta} \mathbb{P}_i$ and $\mathbb{Q}^\zeta := \prod_{i \in [\zeta,\gamma)} \mathbb{P}_i$ (both taken with Easton support) so that

$$\mathbb{Q} \cong \mathbb{Q}_\zeta \times \mathbb{Q}^\zeta$$

Claim 1. *Let $\zeta < \gamma$. Then $\mathbb{Q}_\zeta$ is $(2^{\sup_{i<\zeta} \kappa_i})^+$-cc and $\mathbb{Q}^\zeta$ is $<(2^{\sup_{i<\zeta} \kappa_i})^+$-directed closed.*

[9] Our convention here is that $(2^{\sup_{j<0} \kappa_j})^+ := \mu$.

Proof. Clearly, $|\mathbb{Q}_\zeta| \leq 2^{\sup_{i<\zeta}\kappa_i}$ which implies the statement. $\mathbb{Q}^\zeta$ is an Easton-support product of $<(2^{\sup_{i<\zeta}\kappa_i})^+$-directed closed forcings and thus $<(2^{\sup_{i<\zeta}\kappa_i})^+$-directed closed itself. □

Claim 2. *After forcing with $\mathbb{Q}$, for each $\zeta<\gamma$, $2^{\sup_{j<i}\kappa_j}<\kappa_i$.*

Proof. For each $\zeta<\gamma$, $\mathbb{Q}_\zeta$ has size $\leq 2^{\sup_{i<\zeta}\kappa_i}$ and thus cannot increase the size of the powerset of $\sup_{i<\zeta}\kappa_i$. The following extension by $\mathbb{Q}^\zeta$ does not add any new sequences of ordinals of length $<2^{\sup_{i<\zeta}\kappa_i}$ and thus preserves this. □

Let G be $\mathbb{Q}$-generic. For each $i<\gamma$, let G_i be the $\mathbb{P}_i$-generic filter induced by G. In $V[G_i]$, let $\mathcal{J}_i$ and $\mathcal{B}_i$ witness $\mathsf{LIP}((2^{\sup_{j<i}\kappa_j})^+,\kappa_i)$. In $V[G]$, let $\mathcal{I}_i$ be the ideal generated by $\mathcal{J}_i$. We claim that $(\mathcal{I}_i,\mathcal{B}_i)_{i<\gamma}$ witness $\mathrm{SLIP}(\mu,\langle\kappa_i\mid i<\gamma\rangle)$ in $V[G]$.

Claim 3. *Let $i<\gamma$. Then $\mathcal{I}_i$ is $<\kappa_i$-complete in $V[G]$.*

Proof. By the definition, $\mathcal{I}_i=\{A\in\mathcal{P}^{V[G]}(\kappa_i)\mid\exists B\in\mathcal{J}_i(A\subseteq B)\}$. Let $(A_\alpha)_{\alpha<\theta<\kappa_i}$ be a sequence in $V[G]$ consisting of members of $\mathcal{I}_i$. Without loss of generality, $A_\alpha\in\mathcal{J}_i$ for all $\alpha<\Theta$. Since $V[G]$ is an extension of $V[G_i]$ using the poset $\mathbb{Q}_i\times\mathbb{Q}^{i+1}$ and $\mathbb{Q}^{i+1}$ is $<\kappa_i^+$-closed in V (and thus $<\kappa_i^+$-distributive in $V[G_i]$), it follows that $(A_\alpha)_{\alpha<\theta<\kappa_i}\in V[\prod_{j\leq i}G_j]$. On the other hand, $\mathbb{Q}_i$ is κ_i-cc in $V[G_i]$ by Claim 1, so we can find a sequence $(X_\alpha)_{\alpha<\Theta}\in V[G_i]$ consisting of $X_\alpha\subseteq\mathcal{J}_i^{<\kappa_i}$ such that $A_\alpha\in X_\alpha$. By the $<\kappa_i$-completeness of $\mathcal{J}_i$ in $V[G_i]$, we deduce that $\bigcup_{\alpha<\theta}A_\alpha\in\mathcal{J}_i\subseteq\mathcal{I}_i$. □

The more substantial claim is the following:

Claim 4. *Let $\zeta<\gamma$ and $i\in[\zeta,\gamma)$. Let H^ζ be the $\mathbb{Q}^\zeta$-generic filter induced by G. In $V[H^\zeta]$, $\mathcal{B}_i$ is dense in the complement of the ideal generated by $\mathcal{J}_i$ and $<(2^{\sup_{j<\zeta}\kappa_j})^+$-directed closed.*

Proof. Let $\mathcal{K}_i$ be the ideal generated by $\mathcal{J}_i$ in $V[H^\zeta]$. The model $V[H^\zeta]$ is an extension of $V[G_i]$ using $\prod_{j\in[\zeta,\gamma),j\neq i}\mathbb{P}_i=\prod_{j\in[\zeta,i)}\mathbb{P}_j\times\prod_{j\in(i,\zeta)}\mathbb{P}_j$. The second part adds no subsets of κ_i and thus cannot violate the density of $\mathcal{B}_i$. The first part has size $<\kappa_i$ and thus also does not violate the density of $\mathcal{B}_i$ by the following argument: Suppose τ is a $\prod_{j\in[\zeta,i)}\mathbb{P}_j$-name for an element of $\dot{\mathcal{K}}_i^+$, forced by p. Then there is $q\leq p$ such that the set X_q of all elements of κ_i forced by q to be in τ is in $\mathcal{J}_i^+$ by the $<\kappa_i$-completeness of $\mathcal{J}_i$ and the small size of the poset ($<\kappa_i$). Ergo we can find $X\in\mathcal{B}_i$ such that $X\subseteq X_q$ and thus $q\Vdash\check{X}\subseteq\tau$.

Since $V[H^\zeta]$ is an extension of $V[G_i]$ using a $<(2^{\sup_{j<\zeta}\kappa_j})^+$-closed poset, $\mathcal{B}_i$ is still $<(2^{\sup_{j<\zeta}\kappa_j})^+$-directed closed in that model. □

For $\zeta=0$, we obtain from the previous claim that every $\mathcal{B}_i$ is $<\mu$-directed closed in $V[G]=V[H^0]$. For $\zeta>0$, we argue the distributivity of $(\prod_{i\in[\zeta,\gamma)}\mathcal{B}_i,\subseteq)$ as follows: Fix $\zeta<\gamma$. In $V[H^\zeta]$, the poset

$$\left(\prod_{i\in[\zeta,\gamma)}\mathcal{B}_i,\subseteq\right)$$

is $<(2^{\sup_{j<\zeta}\kappa_j})^+$-directed closed. Since $V[G]$ is an extension of $V[H^\zeta]$ using $\mathbb{Q}_\zeta$ (a poset with the $(2^{\sup_{j<\zeta}\kappa_j})^+$-cc), Easton's Lemma (Lemma 2.10) implies that, in $V[G]$, $\prod_{i\in[\zeta,\gamma)}\mathcal{B}_i$ is still $<(2^{\sup_{j<\zeta}\kappa_j})^+$-distributive. □

Remark 3.10. In Section 4, we will assume that the GCH holds and that our sequence $\langle \kappa_i \mid i < \gamma \rangle$ is **continuous** (i.e. only the κ_{i+1}'s will be measurable). Consequently, in §4 we will be concerned with $\mathsf{LIP}(\kappa_i^{++}, \kappa_{i+1})$-forcing notions.

3.1. **The main forcing.** Our setup assumptions for the rest of the section are:

Setup 3.11. We assume that $\langle \mu \rangle^\frown \langle \kappa_i \mid i < \gamma \rangle$ is an increasing sequence of regular cardinals witnessing $\mathrm{SLIP}(\mu, \langle \kappa_i \mid i < \gamma \rangle)$. (In particular, $\gamma < \mu < \kappa_0$.) We set $\vec{\kappa} = \langle \kappa_i \mid i < \gamma \rangle$, and denote by κ the limit of this sequence. Finally, we denote the witnessing sequence of ideals and dense sets as $\vec{\mathcal{I}}, \vec{\mathcal{B}}$, respectively.

Definition 3.12. Let $\mathbb{P}(\mu, \vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}})$ be the poset consisting of sequences

$$p = \langle p_i \mid i < \gamma \rangle \in \prod_{i<\gamma} (\kappa_i \cup \mathcal{B}_i)$$

whose *support* $\mathrm{supp}(p) := \{ i < \gamma \mid p_i \notin \mathcal{B}_i \}$ is finite.

Given conditions p, q, write $p \leq q$ if $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$ and for each $i < \gamma$:

(1) $p_i = q_i$ if $i \in \mathrm{supp}(q)$;
(2) $p_i \in q_i$ if $i \in \mathrm{supp}(p) \smallsetminus \mathrm{supp}(q)$;
(3) $p_i \subseteq q_i$ otherwise.

We write $p \leq_0 q$ if $p \leq q$ and $\mathrm{supp}(p) = \mathrm{supp}(q)$.

This poset can be viewed as a form of a Laver-style Namba forcing. However, it is defined as a product instead of possessing the form of a tree. This is because of the following problems: On one hand, if we take the version of higher Namba forcing where the trees have height some cardinal κ and any stem of length $< \kappa$, it is unclear whether this forcing has the Prikry property. On the other hand, if we take trees of height some cardinal κ which split at cofinitely many levels, already the first branch of length ω is generic and thus does not have a defined successor set. Thanks to the discreteness of $\vec{\kappa}$ and the "closure properties" of $\vec{\mathcal{I}}$ and $\vec{\mathcal{B}}$ we will be able to identify $(\mathbb{P}(\mu, \vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}}), \leq, \leq_0)$ as a Prikry-type forcing. However, the arguments will be much more technical.

The following is a consequence of members of $\vec{\mathcal{B}}$ being $<\mu$-directed closed:

Lemma 3.13. *$\langle \mathbb{P}(\mu, \vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}}), \leq_0 \rangle$ is $<\mu$-directed closed.* □

Remark 3.14. The $<\mu$-directed-closure of the poset does not play any role in the analysis of this section. It will be critical in Section 4 when we define the iteration yielding Theorem B. Indeed, the directed closure is one of the cruxes of Theorem 4.5, which ensures that the failure of AP is preserved throughout the iteration.

From basic cardinal computations we deduce the following:

Lemma 3.15. *Assume the* GCH. *Then, $\mathbb{P}(\mu, \vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}})$ has the κ^{++}-cc.* □

Now turn to define the minimal extensions of a condition:

Definition 3.16. Let $p \in \mathbb{P}(\mu, \vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}})$, $i \notin \mathrm{supp}(p)$ and $\alpha \in p(i)$. The one-point extension of p by α, $p^\frown(i, \alpha)$, is the sequence $q = \langle q_j \mid j < \gamma \rangle$ defined as follows:

(1) $\mathrm{supp}(q) = \mathrm{supp}(p) \cup \{i\}$.
(2) $q_j := \begin{cases} p_j, & \text{if } j \in \mathrm{supp}(p); \\ \alpha, & \text{otherwise.} \end{cases}$

Given a finite set $I \subseteq \gamma \setminus \mathrm{supp}(p)$ and ordinals $\vec{\alpha} = \langle \alpha_0, \ldots, \alpha_{|I|-1} \rangle \in \prod_{i \in I} p(i)$, the sequence $p^\frown(I, \vec{\alpha})$ is defined by recursion in the obvious manner.

Lemma 3.17. *Let $p \in \mathbb{P}(\mu, \vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}})$, $I \subseteq \gamma \setminus \mathrm{supp}(p)$ and ordinals $\vec{\alpha} \in \prod_{i \in I} p(i)$. Then, $p^\frown(I, \vec{\alpha})$ is a condition in $\mathbb{P}(\mu, \vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}})$ and $p^\frown(I, \vec{\alpha}) \leq p$* □

The next easy observation will be critical later on:

Lemma 3.18. *Let $p \in \mathbb{P}(\mu, \vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}})$ and $I \subseteq \gamma \setminus \mathrm{supp}(p)$ finite. Then,*

$$\{p^\frown(I, \vec{\alpha}) \mid \vec{\alpha} \in \textstyle\prod_{i \in I} p(i)\}$$

is a maximal antichain $\leq$-below p with size $<\kappa_{\max(I)+1}$. □

Convention 3.19. To simplify notations we shall write $\mathbb{P}$ instead of $\mathbb{P}(\mu, \vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}})$.

In our work, the poset $\mathbb{P}$ plays a similar role to that of Cohen forcing in the original Mitchell poset (see [Mit72]): On its own, the Levy-collapse $\mathrm{Coll}(\mu, \kappa^+)$ has the property that every initial segment of the added surjection from μ onto κ^+ lies in the ground model (by the $<\mu$-closure of the poset). However, Mitchell noticed that by using Cohen forcing $\mathrm{Add}(\tau, 1)$ (for $\tau < \mu$) *before* the Levy-collapse, the iteration $\mathrm{Add}(\tau, 1) * \dot{\mathrm{Coll}}(\check{\mu}, \check{\kappa}^+)$ has the opposite property: Every new function from μ into the ordinals has an initial segment which does not lie in the ground model. This is key to his proof that in Mitchell's generic extension – i.e., collapsing a Mahlo cardinal to become μ^+ – AP_μ fails. In our case, we will be obtaining an even stronger statement (for a smaller class of sequences): After forcing with the iteration $\mathbb{P} * \dot{\mathrm{Coll}}(\check{\mu}, \check{\kappa}^+)$, every surjection from μ into κ^+ has an initial segment *which is not even covered by a ground-model set of size $<\kappa$*, an idea that traces back to [JL25]. This is key to show that $\mathbb{P} * \dot{\mathrm{Coll}}(\check{\mu}, \check{\kappa}^+)$ does not make κ^+ d-approachable with respect to any normal coloring d on κ^+ in the ground model. Unlike in [JL25], this will be shown (see Lemma 3.28) as a culmination of a *Prikry analysis* of $\mathbb{P}$.

Let us now prove that $\mathbb{P}$ works as intended. We first have to show that the poset has the *strong Prikry Property* and afterwards we will derive both the Prikry Property and the κ_0-pure decidability property (see Definition 2.12).

One notable difference between this poset and the Laver-style Namba forcing considered in [JL25, §4] is that the latter can eventually decide names for ordinals below any κ_n. (Because any condition will eventually have a sufficiently long stem.) This implies that the poset does not add bounded subsets of κ. However, since we are using conditions of potentially uncountable length we will also be adding new subsets of $\sup_n \kappa_n$ and thus bounded subsets of κ.

Lemma 3.20. *$(\mathbb{P}, \leq, \leq_0)$ has the strong Prikry property; namely, given a condition $p \in \mathbb{P}$ and a dense open set $D \subseteq \mathbb{P}$ there is $q \leq_0 p$ and $I \subseteq \gamma \setminus \mathrm{supp}(q)$ finite such that for each $\vec{\alpha} \in \prod_{i \in I} q(i)$, $q^\frown(I, \vec{\alpha}) \in D$.*

Proof. Given a condition $r \in \mathbb{P}$ we will say that r is *good* (resp. *really good*) if it witnesses the conclusion of the lemma (resp. by taking $q = r$). If r is not good we will say that r is *bad*. Note that if r is bad any $\leq_0$-extension of it is bad as well.

Toward a contradiction, let us assume that the given condition p is bad.

Claim 1. *Given an index $i \notin \mathrm{supp}(p)$ there is $q \leq_0 p$ such that $q \restriction i = p \restriction i$ and $q^\frown(i, \alpha)$ is bad for all $\alpha \in q(i)$.*

Proof of claim. Let W denote the positive set $p(i)$. For each $\alpha \in W$ the set D_α consisting of all $q \in \prod_{j\in(i,\gamma)} \mathcal{B}_j$ such that either $(p \restriction i)^\frown\langle\alpha\rangle^\frown q$ is bad or there is $r \leq_0 p \restriction i$ such that $r^\frown\langle\alpha\rangle^\frown q$ is really good is dense open in $(\prod_{j\in(i,\gamma)} \mathcal{B}_j, \subseteq)$. (Note that this may be false if we replace "really good" by "good".) By $<\kappa_i^+$-distributivity of $(\prod_{j\in(i,\gamma)} \mathcal{B}_j, \subseteq)$, the set $D = \bigcap_{\alpha\in W} D_\alpha$ is dense in $\prod_{j\in(i,\gamma)} \mathcal{B}_j$. Let $q^* \in D$.

We split W into two disjoint sets as follows:

$$W_0 := \{\alpha \in W \mid (p \restriction i)^\frown\langle\alpha\rangle^\frown q^* \text{ is bad}\},$$

$$W_1 := \{\alpha \in W \mid \exists r \leq_0 (p \restriction i)\, (r^\frown\langle\alpha\rangle^\frown q^* \text{ is really good})\}.$$

Suppose that $W_0 \in \mathcal{I}_i^+$. Then, there is $Z \in \mathcal{B}_i$ such that $Z \subseteq W_0$ (by density of $\mathcal{B}_i$). Let q be defined as $(p \restriction i)^\frown Z^\frown q^* \leq_0 p$. Clearly, q is as wished.

Suppose otherwise that $W_1 \in \mathcal{I}_i^+$. For each $\alpha \in W_1$ there is a condition r_α and a finite set $I_\alpha \subseteq \gamma$ witnessing the property. Note that there are at most $2^{\sup_{j<i}\kappa_j}$-many of such r_α's and by assumption $2^{\sup_{j<i}\kappa_j} < \kappa_i$. Additionally, there are at most γ many such I_α's and $\gamma < \kappa_0$. Hence, by $<\kappa_i$-completeness of $\mathcal{I}_i^+$ and density of $\mathcal{B}_i$, there is $Z \subseteq W_1$ in $\mathcal{B}_i$ and r^*, I^* such that $r^* = r_\alpha$ and $I_\alpha = I$ for all $\alpha \in Z$. Let $q = r^{*\frown}\langle Z\rangle^\frown q^*$. Clearly, $q \leq_0 p$. A moment of reflection makes clear that p is good as witnessed by q and $I = I^* \cup \{i\}$. But we were assuming that p was bad, so W_1 cannot be $\mathcal{I}_i$-positive, which rules out this second alternative.

Therefore, it has to be the case that $W_0 \in \mathcal{I}_i^+$ and so q is as wished. □

Without loss of generality, suppose that $\mathrm{supp}(p) = \emptyset$. Now we use the previous claim to survey over all possible extensions of p. More precisely, we define, by induction, a $\leq_0$-decreasing sequence $\langle q_\delta \mid \delta < \gamma\rangle$ in $\mathbb{P}$ with the property that for each $I \in [\delta]^{<\omega}$ and $\vec{\alpha} \in \prod_{i\in I} q_\delta(i)$, $q_\delta{}^\frown(I, \vec{\alpha})$ is bad.

Suppose we have succeeded in defining $\langle q_\delta \mid \delta < \epsilon\rangle$.

Case ϵ is limit: In this case we simply take a $\leq_0$-lower bound q_ϵ for this sequence. Notice that this choice works: Given any $I \in [\epsilon]^{<\omega}$ and $\vec{\alpha} \in \prod_{i\in I} q_\epsilon(i)$ there is an index $\delta < \epsilon$ such that $q_\epsilon{}^\frown(I, \vec{\alpha}) \leq_0 q_\delta{}^\frown(I, \vec{\alpha})$ and the latter condition is bad. By transitivity of the property of being bad, it follows that $q_\epsilon{}^\frown(I, \vec{\alpha})$ is bad as well.

Case ϵ is successor: Suppose that $\epsilon = \bar{\epsilon} + 1$. Let $\{(I_i, \vec{\alpha}_i) \mid i < \theta\}$ be an enumeration of all possible finite subsets of $\bar{\epsilon}$ and $\vec{\alpha}_i \in \prod_{j\in I} q_{\bar{\epsilon}}(j)$.

For each $i < \theta$ let us consider the set of all possible "bad tails":

$$D_i = \{q \in \textstyle\prod_{j\in[\bar{\epsilon},\gamma)} \mathcal{B}_j \mid \forall \beta \in q(\bar{\epsilon})\, ((q_{\bar{\epsilon}} \restriction \bar{\epsilon}^\frown q)^\frown(I_i, \vec{\alpha}_i)^\frown(\bar{\epsilon}, \beta) \text{ is bad})\}.$$

This set is clearly dense in $(\prod_{j\in[\bar{\epsilon},\gamma)} \mathcal{B}_j, \subseteq)$ as per the previous claim.

Note that $\theta \leq \sup_{\eta<\bar{\epsilon}} \kappa_\eta$, since there are at most $|\bar{\epsilon}|$-many finite subsets of $\bar{\epsilon}$ and any $\vec{\alpha}_i$ is a function from I_i into $\sup_{\eta<\bar{\epsilon}} \kappa_\eta$. As a result, we can let $q \in \bigcap_{i<\theta} D_i$, which exists by $<(\sup_{\eta<\bar{\epsilon}} \kappa_\eta)^+$-distributivity. Clearly, $q_\epsilon = (q_{\bar{\epsilon}} \restriction \bar{\epsilon})^\frown q$ is as desired.

After this construction we obtain a $\leq_0$-decreasing sequence $\langle p_\delta \mid \delta < \gamma\rangle$. In the end, we let p^* be a $\leq_0$-lower bound for this sequence (which exists because $\gamma < \mu$). As in the limit case before, $p^{*\frown}(I, \vec{\alpha})$ is bad for every $I \in [\gamma]^{<\omega}$ and $\vec{\alpha} \in \prod_{i\in I} p^*(i)$.

By density of D there is $q \leq p^*$ inside it. Let $(I, \vec{\alpha})$ be such that $q \leq_0 p^{*\frown}(I, \vec{\alpha})$. By the previous comments, $p^{*\frown}(I, \vec{\alpha})$ is bad, and as a result so is q as well. However, $q \in D$, so it must be really good. This yields the desired contradiction. □

The previous proof in fact gives the following *fusion-like* strong Prikry Property:

Lemma 3.21. *For each condition $p \in \mathbb{P}$, $i \in (\max(\mathrm{supp}(p)), \gamma)$ and a dense open set $D \subseteq \mathbb{P}$ there is $q \leq_0 p$ with $q \restriction i = p \restriction i$ and $I \subseteq \gamma \setminus \mathrm{supp}(q)$ finite such that for each $\vec{\alpha} \in \prod_{i \in I} q(i)$, $q^\frown(I, \vec{\alpha}) \in D$.*

Proof sketch. Run the same argument as in the previous lemma with the only difference that in the construction following Claim 1 the conditions q_δ are taken in such a way that they agree up to coordinate i. □

For each $i < \gamma$ let $\leq_0^i$ denote the subordering of $\leq_0$ defined by

$$p \leq_0^i q \Leftrightarrow p \leq_0 q \text{ and } p \restriction i = q \restriction i.$$

Lemma 3.22. *For each $p \in \mathbb{P}$ and $i > \max(\mathrm{supp}(p))$ the poset*

$$\mathbb{P}(p, i) := (\{q \in \mathbb{P} \mid q \leq_0^i p\}, \leq_0)$$

is $<(2^{\sup_{j<i} \kappa_j})^+$-distributive.

Proof. It follows from $(\prod_{j \in [i,\gamma)} \mathcal{B}_j, \subseteq)$ being $<(2^{\sup_{j<i} \kappa_j})^+$-distributive. □

The above has the following important consequence:

Lemma 3.23. *Forcing with $\mathbb{P}$ preserves κ^+.*

Proof. Towards a contradiction, suppose that this is not the case. Let $p \in \mathbb{P}$ be a condition, $\lambda < \kappa$ a regular cardinal and a $\mathbb{P}$-name $\dot{f}$ such that

$$p \Vdash \text{“}\dot{f} \colon \check{\lambda} \to \check{\kappa}^+ \text{ is cofinal”}.$$

Let $i < \gamma$ be an index such that $\kappa_i > \lambda$. For each $\alpha < \lambda$ let

- D_α be the dense set of conditions in $\mathbb{P}$ deciding a value for $\dot{f}(\alpha)$,
- $E_\alpha := \{q \leq_0^{i+1} p \mid \exists I \in [\gamma]^{<\omega}\, \forall \vec{\theta} \in \prod_{i \in I} q(i)\, q^\frown(I, \vec{\theta}) \in D_\alpha\}$.

By Lemma 3.21, E_α is $\leq_0^{i+1}$-dense below p.

Set $\mathbb{Q} := \mathbb{P}(p, i)$. By Lemma 3.22, $\mathbb{Q}$ is $<\kappa_i^+$-distributive.

Claim 2. *There is $\leq_0$-decreasing sequence $\langle p_\alpha \mid \alpha \leq \lambda \rangle$ of conditions in $\mathbb{Q}$ and finite sets $I_\alpha \in [\gamma]^{<\omega}$ such that $p_\alpha{}^\frown(I_\alpha, \vec{\theta}) \in D_\alpha$ for all $\vec{\theta} \in \prod_{i \in I_\alpha} p_\alpha(i)$.*

Proof. Set $\chi := (2^{\sup_{j<i} \kappa_j})$. We utilize Foreman's Theorem 2.9 which asserts that INC does not have a winning strategy in $\mathcal{G}(\mathbb{Q}, \chi+1)$. Suppose that $\langle p_\alpha, q_\alpha \mid \alpha < \beta \rangle$ is a run in this game. Then, COM moves p_β. (If it could not move then INC would have had a winning strategy.) In response, INC plays as follows: If $\beta = \chi$ then $q_\beta := p_\beta$. Alternatively, INC finds $q_\beta \leq_0 p_\beta$ in E_β. □

For each $\alpha < \lambda$ let Δ_α be the set of all possible decisions made by p_α; namely,

$$\Delta_\alpha = \{\beta < \kappa^+ \mid \exists \vec{\theta} \in \textstyle\prod_{i \in I_\alpha} p_\alpha(i)\, (p_\alpha{}^\frown(I_\alpha, \vec{\theta}) \Vdash \dot{f}(\check{\alpha}) = \check{\beta})\}.$$

This set is of cardinality $<\kappa^+$. Therefore, there is $\rho_\alpha < \kappa^+$ such that $\sup \Delta_\alpha < \rho_\alpha$. By regularity of κ^+ (in V) the supremum of all of these ρ_α's (call it ρ) must be below κ^+. We claim that p_λ forces “range$(\dot{f}) \subseteq \check{\rho}$” which will yield a contradiction.

Let $\alpha < \lambda$ and $q \leq p_\lambda$ deciding the value of $\dot{f}(\check{\alpha})$. Without loss of generality, $q \leq p_\alpha{}^\frown(I_\alpha, \vec{\theta})$ for some of such $\vec{\theta}$. Therefore, $q \Vdash \dot{f}(\check{\alpha}) < \check{\rho}_\alpha < \check{\rho}$. By density, p_λ has the sought property, which completes the proof. □

We can derive the Prikry property from the strong Prikry property. Indeed, now we prove that $(\mathbb{P}, \leq, \leq_0)$ has pure $<\kappa_0$ decidability and then derive from it the Prikry property of $(\mathbb{P}, \leq, \leq_0)$.

Lemma 3.24. *Let $p \in \mathbb{P}$ be a condition and τ a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \tau < \check{\theta}$ for some $\theta < \kappa_0$. Then, there is $p^* \leq_0 p$ and $\delta < \kappa_0$ such that $p^* \Vdash_{\mathbb{P}} \tau = \check{\delta}$.*

Proof. Let $D = \{q \in \mathbb{P} \mid q \perp p \vee \exists \delta\, (q \Vdash_{\mathbb{P}} \tau = \check{\delta})\}$. This is clearly dense open, so that there is $p^* \leq_0 p$ and $I \subseteq \gamma$ finite such that $p^{*\frown}(I, \vec{\alpha}) \in D$ for all $\vec{\alpha} \in \prod_{i\in I} p^*(i)$. Suppose that q was chosen so that I is as small as possible.

We claim that $I = \emptyset$ thus establishing the lemma. Suppose otherwise, and set $J := I \setminus \{\max(I)\}$. For each ordinal $\beta \in p^*(\max(I))$ let $f_\beta \colon \prod_{j\in J} p^*(j) \to \theta$ be the function defined as $\vec{\alpha} \mapsto \delta(\vec{\alpha}^\frown\langle\beta\rangle)$ where $\delta(\vec{\alpha}^\frown\langle\beta\rangle)$ is such that

$$p^{*\frown}(I, \vec{\alpha}^\frown\langle\beta\rangle) \Vdash \tau = \check{\delta}(\vec{\alpha}^\frown\langle\beta\rangle).$$

Since $\vec{\mathcal{I}}(\max(I))$ is a $\kappa_{\max(I)}$-complete ideal and there are $<\kappa_{\max(I)}$-many of such functions there is a $\mathcal{I}(\max(I))$-positive set $W \subseteq p^*(\max(I))$ and a function f such that $f_\beta = f$ for all $\beta \in W$. By density of $\mathcal{B}_{\max(I)}$ we can assume that $W \in \mathcal{B}_{\max(I)}$. Let $q \leq_0 p^*$ be the condition obtained after replacing $p^*(\max(I))$ by W. It follows that for every $\vec{\beta}^\frown\langle\alpha\rangle \in \prod_{i\in I} q(i)$, $q^\frown(I, \vec{\beta}^\frown\langle\alpha\rangle) \Vdash \tau = f(\vec{\beta})$. Therefore, already $q^\frown(J, \vec{\beta})$ decides τ, which contradicts the minimality of I. □

Corollary 3.25. *$(\mathbb{P}, \leq, \leq_0)$ has the Prikry property: Given $p \in \mathbb{P}$ and φ a sentence in the forcing language of $\mathbb{P}$ there is $q \leq_0 p$ such that either $q \Vdash_{\mathbb{P}} \varphi$ or $q \Vdash_{\mathbb{P}} \neg\varphi$.*

Proof. Let $p \in \mathbb{P}$ and φ be as before. Consider the $\mathbb{P}$-name σ defined by

$$\sigma := \{(\check{0}, q) \mid q \Vdash_{\mathbb{P}} \varphi\} \cup \{(\check{1}, q) \mid q \Vdash_{\mathbb{P}} \neg\varphi\}.$$

Since $p \Vdash_{\mathbb{P}} \sigma < \check{2}$ Lemma 3.24 yields $q \leq_0 p$ and $i < 2$ such that $q \Vdash_{\mathbb{P}} \sigma = \check{i}$.

We claim that $q \Vdash^i_{\mathbb{P}} \varphi$:[10] Suppose otherwise, and let $r \leq q$ such that $r \Vdash^{1-i}_{\mathbb{P}} \varphi$. Then, $(1-i, r) \in \sigma$, hence $r \Vdash_{\mathbb{P}} \sigma = 1 \check{-} i$. This is impossible in that $q \Vdash_{\mathbb{P}} \sigma = \check{i}$. □

Corollary 3.26. *Forcing with $\mathbb{P}$ does not add new subsets of 2^{κ_0}.*

In particular, forcing with $\mathbb{P}$ preserves cofinalities $\leq\kappa_0$.

Proof. Let $p \in \mathbb{P}$ and σ be a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \sigma \subseteq (2^{\check{\kappa}_0})$. Without loss of generality, $0 \in \mathrm{supp}(p)$, hence $(\mathbb{P}/p, \leq_0)$ is $<(2^{\kappa_0})^+$-distributive.[11] For each $\alpha < 2^{\kappa_0}$ let $D_\alpha = \{q \in \mathbb{P} \mid q \Vdash_{\mathbb{P}} \alpha \in \sigma \vee q \Vdash_{\mathbb{P}} \alpha \notin \sigma\}$, which is $\leq_0$-dense and $\leq_0$-open. By distributivity, $\bigcap_{\alpha<2^{\kappa_0}} D_\alpha$ is also $\leq_0$-dense, so there is $q \leq_0 p$ in that intersection. Consider $a = \{\alpha < 2^{\kappa_0} \mid q \Vdash_{\mathbb{P}} \check{\alpha} \in \sigma\}$. It follows that $q \Vdash_{\mathbb{P}} \sigma = \check{a}$. □

3.2. **The non-approachable collapse.** The goal now is to prove that if $\theta \in (\gamma, \kappa_0)$ is regular and $\rho := (\kappa^+)^V$, then $\mathbb{P} * \dot{\mathrm{Col}}(\theta, \rho)$ forces that ρ is not d-approachable for every coloring $d \colon [\kappa^+]^2 \to \gamma$ that is normal and sub additive from the perspective of the ground model. This is the critical observation to answer Question 1.1.

Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a forcing poset. One can naturally define a "Prikry order" $\leq_0$ on the two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ by stipulating that

$$(q, \dot{d}) \leq_0 (p, \dot{c}) \text{ if and only if } q \leq_0 p \text{ and } q \Vdash_{\mathbb{P}} \dot{d} \leq_{\dot{\mathbb{Q}}} \dot{e}.$$

Lemma 3.27. *The poset $(\mathbb{P} * \dot{\mathbb{Q}}, \leq, \leq_0)$ has the strong Prikry Property; namely, given $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ and a dense open set $D \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ there is $(q, \dot{d}) \leq_0 (p, \dot{c})$ and $I \subseteq \gamma \setminus \mathrm{supp}(q)$ finite such that for each $\vec{\alpha} \in \prod_{i\in I} q(i)$, $(q^\frown(I, \vec{\alpha}), \dot{d}) \in D$.*

[10] Here the convention is that $q \Vdash^0_{\mathbb{P}} \varphi$ is a shorthand for $q \Vdash_{\mathbb{P}} \varphi$, whereas $q \Vdash^1_{\mathbb{P}} \varphi$ means $q \Vdash_{\mathbb{P}} \neg\varphi$.

[11] Again, this is because our SLIP$(\mu, \langle\kappa_i \mid i < \gamma\rangle)$ assumption.

Proof. Look at $E := \{q \in \mathbb{P} \mid \exists \dot{d}\,(q, \dot{d}) \leq (p, \dot{c}) \wedge (q, \dot{d}) \in D\}$. Clearly, E is dense open for $\mathbb{P}$. By the strong Prikry Property applied to this poset we find $q \leq_0 p$ and $I \subseteq \gamma \setminus \mathrm{supp}(q)$ finite such that for each $\vec{\alpha} \in \prod_{i\in I} q(i)$, $q^\frown(I, \vec{\alpha}) \in E$. This means that for each $\vec{\alpha}$ there is a $\mathbb{P}$-name $\dot{d}_{\vec{\alpha}}$ such that $(q^\frown(I, \vec{\alpha}), \dot{d}_{\vec{\alpha}}) \in D$. Since $\{q^\frown(I, \vec{\alpha}) \mid \vec{\alpha} \in \prod_{i\in I} q(i)\}$ forms a maximal antichain below q we can use the *mixing lemma* so as to produce a $\mathbb{P}$-name $\dot{d}$ such that $q^\frown(I, \vec{\alpha}) \Vdash \dot{d} = \dot{d}_{\vec{\alpha}}$. A moment's reflection makes clear that $(q, \dot{d})$ is the sought condition. □

Lemma 3.28 (Key Lemma)**.** *Let $\dot{\mathbb{C}}$ be a $\mathbb{P}$-name for a γ^+-closed poset. Let $\dot{F}$ be a $\mathbb{P} * \dot{\mathbb{C}}$-name for a cofinal function from $\check{\theta}$ into $\check{\kappa}^+$, where $\theta \in (\gamma, \kappa_0)$ regular, forced by a condition $(p, \dot{c}) \in \mathbb{P} * \dot{\mathbb{C}}$. Then there is $(q, \dot{d}) \leq_0 (p, \dot{c})$ and $\zeta < \theta$ such that for no $(r, \dot{e}) \leq (q, \dot{d})$ and $x \in V$ with $|x| < \kappa$, $(r, \dot{e}) \Vdash \mathrm{range}(\dot{F} \restriction \check{\zeta}) \subseteq \check{x}$.*

Proof. The next is the key claim. Roughly speaking it shows that for any condition $(p', \dot{c}')$ and $i \notin \mathrm{supp}(p')$ there is a $\leq_0$-extension $(p'', \dot{c}'')$ and $\zeta < \theta$ such that the "canonical" maximal antichain at coordinate i, $\{(p''^\frown(i, \beta), \dot{c}'') \mid i \in p''(i)\}$, decides at least κ_i-many different possibilities for $\dot{F}(\zeta)$. Establishing the existence of such a condition which has that property for every i will show that $\mathrm{range}(\dot{F})$ cannot be covered by a ground model set of size $<\kappa$. To simplify notations later on, let us assume that the initial condition $(p, \dot{c})$ is just the trivial condition in $\mathbb{P} * \dot{\mathbb{C}}$.

Claim 3. *Let $(p', \dot{c}') \in \mathbb{P} * \dot{\mathbb{C}}$ and $i \notin \mathrm{supp}(p')$. Then, there is $(p'', \dot{c}'') \leq_0 (p', \dot{c}')$, $\zeta < \theta$ and a sequence $\langle A_\beta \mid \beta \in p''(i)\rangle$ of disjoint sets such that for each $\beta \in p''(i)$,*

$$(p''^\frown(i, \beta), \dot{c}'') \Vdash \dot{F}(\zeta) \in \check{A}_\beta.$$

Proof of claim. The claim is established after combining the strong Prikry Property of $\mathbb{P} * \dot{\mathbb{Q}}$ (Lemma 3.27) with the $<\kappa_i^+$-distributivity of $\mathbb{B} = (\prod_{j\in(i,\gamma)} \mathcal{B}_j, \supseteq)$.

Towards a contradiction, suppose that the claim was false. We will exhibit a winning strategy for INC in the completeness game $\mathcal{G}(\mathbb{B}, \kappa_i + 1)$ (see Definition 2.8). By Foreman's theorem (see Theorem 2.9), this will yield a contradiction with $\mathbb{B}$ being $<\kappa_i^+$-distributive.

Suppose we have constructed $\langle (A_\beta, \zeta_\beta, r_\beta, \langle p_\beta, q_\beta\rangle, \dot{c}_\beta) \mid \beta < \beta^*\rangle$ such that:

- $A_\beta \subseteq \kappa^+$ of size $<\kappa$ and $A_\beta \cap A_{\beta'} = \emptyset$ for $\beta \neq \beta'$.
- ζ_β is an ordinal below θ.
- $r_\beta \leq_0 p' \restriction i$. (I.e., r_β is the "low part" of a condition in $\mathbb{P}$.)
- $\langle p_\beta, q_\beta\rangle$ is a move in the game $\mathcal{G}(\mathbb{B}, \kappa_i + 1)$.
- $(r_\beta^\frown \langle\beta\rangle^\frown q_\beta, \dot{c}_\beta) \Vdash \dot{F}(\check{\zeta}_\beta) \in \check{A}_\beta$.

Assume that $\langle (p_\beta, q_\beta) \mid \beta < \beta^*\rangle$ has been already played in $\mathcal{G}(\mathbb{B}, \kappa_i + 1)$. First COM moves[12] p_{β^*} and let INC makes its move. The second player INC moves as follows. First, since $((p' \restriction i)^\frown\langle\beta^*\rangle^\frown p_{\beta^*}, \dot{c}')$ forces "$\dot{F}\colon \check{\theta} \to \check{\kappa}^+$ is cofinal", this condition itself forces "$\exists \zeta < \check{\theta}\,(\dot{F}(\zeta) \notin \bigcup_{\beta<\beta^*} \check{A}_\beta)$" (note that this union has cardinality $<\kappa^+$).

Let $(p'_{\beta^*}, \dot{c}'_{\beta^*}) \leq_0 ((p' \restriction i)^\frown\langle\beta^*\rangle^\frown p_{\beta^*}, \dot{c}')$ decide this ordinal $\zeta < \theta$ – denote it ζ_{β^*}. It follows that $(p'_{\beta^*}, \dot{c}'_{\beta^*}) \Vdash \dot{F}(\check{\zeta}_{\beta^*}) \notin \bigcup_{\beta<\beta^*} \check{A}_\beta$. By Lemma 3.27 there is a further pure extension $(p''_{\beta^*}, \dot{c}_{\beta^*}) \leq_0 (p'_{\beta^*}, \dot{c}'_{\beta^*})$ and a finite set $I \subseteq \gamma \setminus \mathrm{supp}(p''_{\beta^*})$ such that

[12] If COM could not move then we would have already described a winning strategy for INC.

for every $\vec{\alpha} \in \prod_{i\in I} p''_{\beta^*}(i)$, $(p''_{\beta^*}{}^\frown(I,\vec{\alpha}), \dot{c}_{\beta^*})$ decides the value of $\dot{F}(\check{\zeta}_{\beta^*})$. Note that the set of decisions

$$A_{\beta^*} := \{\eta < \kappa^+ \mid \exists \vec{\alpha} \in \textstyle\prod_{i\in I} p''_{\beta^*}(i)\, (p''_{\beta^*}{}^\frown(I,\vec{\alpha}), \dot{c}_{\beta^*}) \Vdash \dot{F}(\check{\zeta}_{\beta^*}) = \check{\eta}\}$$

is of size $<\kappa$ because so is the antichain $\{p''_{\beta^*}{}^\frown(I,\vec{\alpha}) \mid \vec{\alpha} \in \prod_{i\in I} p''_{\beta^*}(i)\}$. Also,

$$A_{\beta^*} \cap \textstyle\bigcup_{\beta<\beta^*} A_\beta = \emptyset.$$

After this construction, we set $r_{\beta^*} := p''_{\beta^*} \restriction i$ and let INC play $q_{\beta^*} = p''_{\beta^*} \setminus i+1$.

We claim that the above produces a winning strategy for INC in $\mathcal{G}(\mathbb{B}, \kappa_i + 1)$. Suppose that this was not the case and that $\{(p_\beta, q_\beta) \mid \beta < \kappa_i\}$ was a winning play for COM in the game $\mathcal{G}(\mathbb{B}, \kappa_i + 1)$. Let p^* be a lower bound for this sequence. Since $\theta < \kappa_0$ and $2^{\sup_{j<i}\kappa_j} < \kappa_i$ there is $W \subseteq p'(i)$ in $\mathcal{B}_i$, $\zeta^* < \theta$ and r^* a low part such that $\zeta_\beta = \zeta^*$ and $r_\beta = r^*$ for all $\beta \in W$. Let $p'' := r^*{}^\frown\langle W\rangle{}^\frown p^*$. Let $\dot{c}''$ be the mixing name of the $\mathbb{P}$-names $\{\dot{c}_\beta \mid \beta \in W\}$. By construction, it follows that $(p'', \dot{c}'') \leq_0 (p', \dot{c}')$ witnesses together with $(\zeta, \langle A_\beta \mid \beta \in W\rangle)$ the conclusion of the claim. Since we were assuming that no such $\leq_0$-extension of $(p', \dot{c}')$ exists it must be the case that COM cannot form such a lower bound, p^*. In turn, this shows that the above procedure describes a winning strategy for INC in $\mathcal{G}(\mathbb{B}, \kappa_i + 1)$. However, this is impossible by Foreman's theorem, which yields the final contradiction. □

Next we use the previous claim to complete the proof of the lemma. By induction define a $\leq_0$-sequence of conditions $\langle (p_i, \dot{c}_i) \mid i < \gamma\rangle$ witnessing the previous lemma relative to sets $\langle A^i_\beta \mid \beta \in p_i(i)\rangle$. At limit stages we take a $\leq_0$-lower bound to the previously constructed sequences (which exists in that $\dot{\mathbb{C}}$ is forced to be γ^+-closed) and then apply Claim 3.[13] Let $(q, \dot{d})$ be a $\leq_0$-lower bound for this sequence and set $\zeta := \sup_{i<\gamma} \zeta_i$. Since $\gamma < \theta$ and θ is regular it follows that $\zeta < \theta$.

Suppose towards a contradiction that there is $x \in V$, $|x| < \kappa$ and a condition $(r, \dot{e}) \leq (q, \dot{d})$ forcing "$\mathrm{range}(\dot{F} \restriction \check{\zeta}) \subseteq \check{x}$". Let $i < \gamma$, $i \notin \mathrm{supp}(r)$, such that $|x| < \kappa_i$. Then, since $r(i)$ is a κ_i-sized subset of $p_i(i)$ there must be some $\beta \in r(i)$ such that $x \cap A^i_\beta = \emptyset$. However, by our construction,

$$(r^\frown(i,\beta), \dot{e}) \leq (p_i{}^\frown(i,\beta), \dot{c}_i) \Vdash \dot{F}(\check{\zeta}_i) \in \check{A}^i_\beta,$$

and by assumption, $(r^\frown(i,\beta), \dot{e}) \vdash \dot{F}(\check{\zeta}_i) \in \check{x}$, a contradiction. □

Corollary 3.29. *Let $\theta \in (\gamma, \kappa_0)$ be any regular cardinal. Then, after forcing with $\mathbb{P} * \dot{\mathrm{Coll}}(\check{\theta}, \check{\kappa}^+)$, κ^+ is not d-approachable with respect to any $d\colon [\kappa^+]^2 \to \mathrm{cf}(\kappa)$ that the ground model thinks as a normal, transitive coloring.*[14]

Proof. Suppose that κ^+ was forced by $\mathbb{P} * \dot{\mathrm{Coll}}(\check{\theta}, \check{\kappa}^+)$ to be d-approachable. Then, there is a condition $(p, \dot{c})$, a name $\dot{A}$ and an ordinal $j < \mathrm{cf}(\kappa)$ such that $(p, \dot{c})$ forces "$\dot{A}$ is unbounded in κ^+" and "$\forall \beta_0, \beta_1 \in \dot{A}\, (\beta_0 < \beta_1 \to \check{d}(\beta_0, \beta_1) \leq \check{j})$" (see Fact 2.27.) Let $\dot{F}\colon \check{\theta} \to \check{\kappa}^+$ be a $\mathbb{P} * \dot{\mathrm{Coll}}(\check{\theta}, \check{\kappa}^+)$-name for the increasing enumeration of $\dot{A}$. By Lemma 3.28 there is $(q, \dot{d}) \leq_0 (p, \dot{c})$ and $i < \theta$ such that for no $(r, \dot{e}) \leq (q, \dot{d})$ and $x \in V$ with $|x| < \kappa$, $(r, \dot{e}) \Vdash \mathrm{range}(\dot{F} \restriction \check{i}) \subseteq \check{x}$. By extending $(q, \dot{d})$ is necessary we may in addition assume that it decides the value of $\dot{F}(\check{i})$ – say this is ξ. We

[13] Note that in this process we can make sure that the witnessing ordinals $\zeta_i < \theta$ increase.

[14] Crucially, the regularity of θ is preserved after forcing with $\mathbb{P}$ thanks to Corollary 3.26.

then deduce that $(q,\dot{d})$ forces "$\mathrm{range}(\dot{F}\restriction\check{i})\subseteq\{\beta<\check{\kappa}^+\mid \check{d}(\beta,\check{\xi})\leq\check{j}\}$"[15] but the latter set is, by normality of $d\in V$, a set in V of size $<\kappa$. A contradiction. □

Remark 3.30. At the beginning of this section we fixed a regular cardinal μ such that $\mu\in(\gamma,\kappa_0)$. We mentioned in Remark 3.14 that the importance of this parameter will not be appreciate until we arrive at Section 4 – this is why we omitted it during the discussion we had so far. Nonetheless, note that both Lemma 3.28 and Corollary 3.29 apply for the particular choice $\theta:=\mu$.

3.3. **On Shelah's problem.** In this section we apply the analysis of the previous section to answer Shelah's Question 1.1. The proof is inspired by Shelah's classical proof from [She79], which can be also found in [Eis10, Theorem 3.44].

Theorem 3.31. *Assume the* GCH *holds. Let λ be a supercompact cardinal, $\gamma<\lambda$ a limit ordinal and $\langle\delta_i\mid i<\gamma\rangle$ an increasing sequence of measurable cardinals above λ with limit δ. Let $\mu\in(\mathrm{cf}(\gamma),\lambda)$ be regular. Then, there is a generic extension preserving regular cardinals $\leq\mu$, where $\delta=\lambda^{+\gamma}$ and $\lambda^{+\gamma+1}\cap\mathrm{cof}(\mu)\notin I[\lambda^{+\gamma+1}]$.*

Proof. Let λ, μ and $\langle\delta_i\mid i<\gamma\rangle$ be as above. By preparing the universe we may assume that the supercompactness of λ is indestructible under λ-directed-closed forcings. To avoid potential conflicts with the application of Fact 2.29 we do make the supercompactness of λ indestructible only under λ-directed-closed forcings that preserve the GCH. This forcing itself preserves the GCH pattern from the ground model [PRS23, §8.1]. Next force with the Easton support product of Levy-collapses

$$\mathrm{Coll}(\lambda,<\delta_0)\times\textstyle\prod_{1\leq i<\gamma}\mathrm{Coll}((\sup_{j<i}\delta_j)^{++},<\delta_i).$$

Each of these posets is $\mathsf{LIP}((\sup_{j<i}\delta_j)^{++},<\delta_i)$ by virtue of the measurability of the δ_i's. Hence, by Lemma 3.9, $\mathrm{SLIP}(\lambda,\langle\delta_i\mid i<\gamma\rangle)$ holds in the resulting generic extension. In particular, $\mathrm{SLIP}(\mu,\langle\delta_i\mid i<\gamma\rangle)$ holds as well. Since λ was made indestructible it remains supercompact. Also, δ becomes $\lambda^{+\gamma}$.

Assume that the above model is our ground model, V. Now, we use the supercompactness of λ to "reflect" the failure of approachability at δ^+. More precisely:

Claim 1. *Let $d\colon[\delta^+]^2\to\mathrm{cf}(\gamma)$ be a transitive normal coloring in V. Then, there is a singular $\kappa\in(\mu,\lambda)$ and a stationary set $S\subseteq E^{\delta^+}_{\kappa^+}$ such that the following hold:*

1. *There is an increasing sequence of regular cardinals $\langle\mu\rangle^\frown\langle\kappa_i\mid i<\gamma\rangle$ with $\sup_{i<\gamma}\kappa_i=\kappa$ witnessing $\mathrm{SLIP}(\mu,\langle\kappa_i\mid i<\gamma\rangle)$.*
2. *For each $\alpha\in S$, there is a transitive normal coloring $e\colon[\kappa^+]^2\to\mathrm{cf}(\gamma)$ such that, for each set-sized forcing poset $\mathbb{P}$,*

$$\Vdash_{\mathbb{P}}\text{"}\alpha\text{ is }d\text{-approachable}\Leftrightarrow(\kappa^+)^{\check{V}}\text{ is }e\text{-approachable"}.$$

Proof of claim. Let us consider the following set

$$E=\{\rho<\delta^+\mid(1)_\rho\wedge(2)_\rho\wedge(3)_\rho\}$$

where

- $(1)_\rho$: $\mathrm{cof}(\rho)<\lambda$ and $\mathrm{cof}(\rho)$ is a successor of a singular cardinal of cofinality $\mathrm{cf}(\gamma)$.
- $(2)_\rho$: There is an increasing sequence of regular cardinals $\langle\rho_i\mid i<\gamma\rangle$ converging to $\mathrm{cof}(\rho)^-$, $\mu<\rho_0$, and $\mathrm{SLIP}(\mu,\langle\rho_i\mid i<\gamma\rangle)$ holds.

[15] Here we use that d is transitive; namely, that $d(\beta_0,\xi)\leq d(\beta_0,\beta_1)$ for all $\beta_0<\beta_1<\xi$.

$(3)_\rho$: There is a normal transitive coloring $e\colon [\mathrm{cof}(\rho)]^2 \to \mathrm{cf}(\gamma)$ such that for each set-sized forcing $\mathbb{P}$,

$$\Vdash_{\mathbb{P}} \text{“}\rho \text{ is } d\text{-approachable} \Leftrightarrow \mathrm{cof}(\rho) \text{ is } e\text{-approachable”}$$

We claim that E is stationary.

To show this, fix $j\colon V \to M$ be a δ^+-supercompact embedding and let $C \subseteq \delta^+$ be an arbitrary club. We show that C intersects E. Set $\rho := \sup(j[\delta^+]) \in j(C)$ and note that the following properties hold on the *M-side* of j:

(1) Clearly, $\mathrm{cf}(\rho) = \delta^+$. So, $\mathrm{cf}(\rho) \in (j(\mu), j(\lambda))$ and it is the successor of a singular with cofinality $\mathrm{cf}(\gamma)$.
(2) The predecessor of $\mathrm{cf}(\rho)$ is the limit of the increasing sequence $\langle \delta_i \mid i < \gamma\rangle$ of regular cardinals, $\mu < \delta_0$, and $\mathrm{SLIP}(\mu, \langle \delta_i \mid i < \gamma\rangle)$ holds.
(3) The coloring $e\colon [\mathrm{cf}(\rho)]^2 \to \mathrm{cf}(\gamma)$ defined by
$$e(\alpha, \beta) := j(d)(j(\alpha), j(\beta))$$
is normal and transitive. In addition for each set-sized poset $\mathbb{P} \in M$
$$\Vdash_{\mathbb{P}} \text{“}\rho \text{ is } j(d)\text{-approachable} \Leftrightarrow \mathrm{cf}(\rho) \text{ is } e\text{-approachable”}$$

Clauses (1) and (2) are clear, so we just check (3): First e is transitive just because so is $j(d)$. For normality, letting $\alpha < \mathrm{cf}(\rho)$ and $i < \mathrm{cf}(\gamma)$ note that

$$|\{\beta < \alpha \mid e(\beta, \alpha) \leq i\}| = |\{\beta < \alpha \mid j(d)(j(\beta), j(\alpha)) \leq i\}| = |\{\beta < \alpha \mid d(\beta, \alpha) \leq i\}|.$$

Hence,

$$\sup_{\alpha < \mathrm{cf}(\rho)} |\{\beta < \alpha \mid e(\beta, \alpha) \leq i\}| = \sup_{\alpha < \delta^+} |\{\beta < \alpha \mid d(\beta, \alpha) \leq i\}| < \delta = \mathrm{cf}(\rho)^-,$$

by normality of d.

Let $\mathbb{P} \in M$ be any set-sized poset and work in $V[\mathbb{P}]$. Then, ρ is $j(d)$-approachable if and only if $j(d)$ is bounded on an unbounded subset of $j[\delta^+]$. (The left-to-right implication is Fact 2.28, and the converse is just trivial.) In turn, $j(d)$ is bounded on an unbounded subset of $j[\delta^+]$ if and only if e is bounded on an unbounded subset of δ^+ by the very definition of e. More formally, if $j(d)$ is bounded on an unbounded set $A \subseteq j[\delta^+]$ with value i, e is bounded on $j^{-1}[A] \subseteq \delta^+$ unbounded with value i. If e is bounded on an unbounded set $A \subseteq \delta^+$ with value i, $j(d)$ is bounded on $j[A] \subseteq j[\delta^+]$ unbounded with value i.

The above argument shows that $\rho \in j(E) \cap j(C)$. Thus, E is stationary. By Fodor's Lemma, there is $S \subseteq E$ stationary where the map $\rho \mapsto \mathrm{cof}(\rho)$ takes a constant value $< \lambda$. This constant value is, by definition of E, a successor of a singular cardinal $\kappa < \lambda$. Clearly, S and κ witness the required properties. □

Let $\vec{\mathcal{I}}, \vec{\mathcal{B}}$ be witnesses for $\mathrm{SLIP}(\mu, \langle \kappa_i \mid i < \gamma\rangle)$ and $G * \dot{C} \subseteq \mathbb{P}(\mu, \vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}}) * \dot{\mathrm{Coll}}(\check{\mu}, \check{\kappa}^+)$ generic. This forcing does not disturb the GCH pattern of V, so Fact 2.29 still applies in $V[G * \dot{C}]$. With this in mind we prove the following:

Claim 2. $V[G * \dot{C}] \models \text{“}\delta^+ \cap \mathrm{cof}(\mu) \notin I[\delta^+]\text{”}$.

Proof of claim. Suppose towards a contradiction that the claim was false. In the generic extension, S is a stationary subset of $\delta^+ \cap \mathrm{cof}(\mu)$ because the forcing yielding $V[G * \dot{C}]$ is small relative to δ^+. We next show that $S \notin I[\delta^+]$ holds in $V[G * \dot{C}]$.

First, $d\colon [\delta^+]^2 \to \mathrm{cf}(\gamma)$ remains a normal transitive coloring in $V[G * \dot{C}]$ in that d had this property in V and $|\mathbb{P}(\mu, \vec{\kappa}, \vec{\mathcal{I}}, \vec{\mathcal{B}}) * \mathrm{Col}(\check{\mu}, \check{\kappa}^+)| < \delta$. As a result,

$(\delta^+ \cap \mathrm{cof}(\leq\gamma)) \cup S(d)^{V[G*\dot{C}]}$ generates $I[\delta^+]$ (Fact 2.29). Therefore, we will be done if we show that $S \cap S(d)^{V[G*\dot{C}]} = \emptyset$ holds. By our previous analysis, for each $\alpha \in S$, there is a normal transitive coloring $e\colon [\kappa^+]^2 \to \mathrm{cf}(\gamma)$ in V such that α is d-approachable in $V[G*\dot{C}]$ if and only if that $(\kappa^+)^V$ is e-approachable in $V[G*\dot{C}]$.[16] Invoking Corollary 3.29 for $\theta := \mu$ we conclude that this latter condition is false and therefore that no $\alpha \in S$ is d-approachable in $V[G * \dot{C}]$. □

The above completes the proof of the theorem. □

Remark 3.32. In [She94, Problem 2.5, p.439], Shelah also asked whether there can be stationary sets, outside the approachability ideal, consisting of ordinals with inaccessible cofinality. Theorem 3.31 also answers this question in the affirmative.

The next corollary generalizes the main result of Jakob–Levine [JL25], where the authors solved Question 1.1 for the case $\kappa = \aleph_\omega$.

Corollary 3.33. *Assume the* GCH *holds. Let λ be a supercompact cardinal, $\aleph_\gamma$ a singular cardinal and $\langle \delta_i \mid i < \gamma\rangle$ an increasing sequence of measurable cardinals above λ with limit δ. Let $\mu \in (\mathrm{cf}(\gamma), \aleph_\gamma)$ be regular. Then, there is a generic extension preserving regular cardinals $\leq\mu$ where $\aleph_{\gamma+1} \cap \mathrm{cof}(\mu) \notin I[\aleph_{\gamma+1}]$.*

Proof. Let V denote the generic extension constructed in Theorem 3.31 and S the stationary set $\lambda^{+\gamma+1} \cap \mathrm{cof}(\mu)$. Force with the Levy collapse $\mathbb{C} := \mathrm{Coll}(\mu^+, \lambda)$. In the generic extension $V[\mathbb{C}]$, we have $\lambda^{+\gamma+1} = \aleph_{\gamma+1}$, S is stationary and we claim that $S \cap S(d) = \emptyset$ still holds. Assume for a contradiction that $S \cap S(d)^{V[\mathbb{C}]}$ is non-empty.

Let $\alpha \in S \cap S(d)^{V[\mathbb{C}]}$ be arbitrary. By Fact 2.28, for every unbounded subset $A \subseteq \alpha$ with $\mathrm{otp}(A) = \mu$, there exists an unbounded subset $B \subseteq A$ on which d is bounded. Since $\mathrm{otp}(B) = \mu$ and $\mathbb{C}$ is $<\mu^+$-closed, it follows that $B \in V$. This implies that $\alpha \in S \cap S(d)^V$, contradicting our assumption in the ground model. Hence, $S \cap S(d)^{V[\mathbb{C}]}$ is empty and thus, by Fact 2.29, $S \notin I[\aleph_{\gamma+1}]$. □

3.4. **On a model of Gitik and Sharon.** In [GS08], Gitik–Sharon constructed a model in which $\aleph_{\omega^2}$ is strong limit, and both the Singular Cardinal Hypothesis (SCH) and AP fail at $\aleph_{\omega^2}$. It turns out that in the model of [GS08], one cannot predetermine the cofinality of the non-approachable stationary set $S \subseteq \aleph_{\omega^2+1}$. The obstacle, as usual, is that this cofinality is determined by $\mathrm{cf}(\sup(M \cap \kappa^{+\omega+1}))$, where $M \prec H(\Theta)$ is a κ-Magidor model[17] and κ is a $\kappa^{+\omega+2}$-supercompact cardinal.

In the next theorem, we generalize Gitik–Sharon's result by showing that the failure of SCH is consistent with the existence of a non-approachable stationary set of any prescribed cofinality. More precisely, we show the following:

Theorem 3.34. *Assume the* GCH *holds. Let λ be a supercompact cardinal, $\gamma < \lambda$ a limit ordinal and assume that there is an increasing sequence of length γ above λ consisting of measurable cardinals. Then, for each regular cardinal $\mu \in (\mathrm{cf}(\gamma), \aleph_{\gamma^2})$, there is a generic extension where $\aleph_{\gamma^2+1} \cap \mathrm{cof}(\mu) \notin I[\aleph_{\gamma^2+1}]$ and $\mathrm{SCH}_{\aleph_{\gamma^2}}$ fails.*

Proof. Let $d\colon [\delta^+]^2 \to \gamma$ be a normal transitive coloring, where δ is the supremum of the aforementioned sequence of measurable cardinals. By the argument in Theorem 3.31, there is a generic extension in which λ remains supercompact and there

[16] Note that $e\colon [\kappa^+]^2 \to \mathrm{cf}(\gamma)$ is not normal in $V[G * \dot{C}]$ as $(\kappa^+)^V$ is collapsed.

[17] In [GS08], the authors refer to these models as *supercompact models*. The terminology *κ-Magidor model* is borrowed from [MV21b].

exists a stationary set $S \subseteq \lambda^{+\gamma+1} \cap \mathrm{cf}(\mu)$ consisting of non-d-approachable points. Working over this generic extension we first make λ indestructible under λ-directed-closed forcings (see [Lav78]), and then force with Cohen forcing $\mathrm{Add}(\lambda, \lambda^{+\omega+2})$. In the resulting extension λ remains supercompact. We can arrange this latter two-step iteration to be $<\mu^+$-directed-closed and λ^+-cc. As a result, S remains stationary and no new d-approachable points of cofinality μ are added. Thus, $S \cap S(d) = \emptyset$ also holds in this model. For simplicity, denote this extension by V.

Building upon work of Gitik–Sharon, Sinapova showed in her Ph.D. thesis [Sin08] that in V there exists a Mitchell-increasing sequence $\vec{\mathcal{U}} = \langle \mathcal{U}_\xi \mid \xi < \gamma \rangle$ of normal fine ultrafilters on $\mathcal{P}_\lambda(\lambda^{+\xi})$, together with a sequence of *guiding generics* $\vec{K} = \langle K_\xi \mid \xi < \gamma \rangle$ for the Levy collapse $\mathrm{Coll}(\lambda^{+\gamma+2}, <j_{\mathcal{U}_\xi}(\lambda))^{\mathrm{Ult}(V,\mathcal{U}_\xi)}$. Using this, she defined a Magidor-style variant of the Gitik–Sharon forcing, denoted $\mathbb{S}(\vec{\mathcal{U}}, \vec{K})$.

$\mathbb{S}(\vec{\mathcal{U}}, \vec{K})$ satisfies the following properties:

(1) $\mathbb{S}(\vec{\mathcal{U}}, \vec{K})$ is $\lambda^{+\gamma+1}$-cc. (See [Sin08, §3]).
(2) $\mathbb{S}(\vec{\mathcal{U}}, \vec{K})$ collapses all cardinals in the interval $(\lambda, \lambda^{+\gamma+1})$. (See [Sin08, §3])
(3) $\mathbb{S}(\vec{\mathcal{U}}, \vec{K})$ forces "$\lambda = \dot{\kappa}_0^{+\gamma^2}$", where $\dot{\kappa}_0$ is the name for the first *Prikry point*.
(4) $\langle \mathbb{S}(\vec{\mathcal{U}}, \vec{K}), \leq, \leq_0 \rangle$ has the strong Prikry Property (See [Pov20, Lemma 3.18].)[18]
(5) There is a dense subposet of $\mathbb{S}(\vec{\mathcal{U}}, \vec{K})$ whose $\leq_0$-ordering is $<\mu^+$-closed.[19]

To simplify notations, let us denote the above dense subposet $\mathbb{S}$.

Claim 3. $\Vdash_{\mathbb{S}}$ "$\check{S} \subseteq \lambda^{+\gamma+1} \cap \mathrm{cof}(\mu)$ *is stationary and* $\check{S} \cap \dot{S}(\check{d}) = \emptyset$".

Proof of claim. Clearly, S remains stationary by the chain condition of the forcing. The reason why all points in S retain cofinality μ is essentially the same as that explaining why $S \cap S(d)^{V[S]} = \emptyset$. Thus, we provide details only for the latter part.

It suffices to prove that for $\alpha < \lambda^{+\gamma+1}$ with $\mathrm{cf}(\alpha) = \mu$, the truth value of "$\alpha$ is not d-approachable" is not changed by forcing with $\mathbb{S}(\vec{\mathcal{U}}, \vec{K})$. So let $\alpha < \lambda^{+\gamma+1}$ have cofinality μ. Let $A \subseteq \alpha$ be unbounded with ordertype μ. By Fact 2.28, after forcing with $\mathbb{S}(\vec{\mathcal{U}}, \vec{K})$, α is d-approachable if and only if d is bounded on an unbounded subset of A. However, since A has size μ in the ground model and $\mathbb{S}(\vec{\mathcal{U}}, \vec{K})$ does not add any new subsets of μ by point (5), α clearly remains non-d-approachable after forcing with $\mathbb{S}(\vec{\mathcal{U}}, \vec{K})$. □

Let $G \subseteq \mathbb{S}$ be a generic filter. In $V[G]$, μ is a regular cardinal, $\lambda = (\kappa_0^{+\gamma^2})^V$ and $\lambda^+ = (\lambda^{+\gamma+1})^V$. (Here $\kappa_0 = (\dot{\kappa}_0)_G$.) Now, force with $\mathrm{Coll}(\mu^+, <\kappa_0)$. Let H be a generic filter for this poset. In $V[G * H]$, κ_0 becomes μ^+ and $\lambda = \aleph_{\gamma^2}$ because orginally $\mu < (\aleph_{\gamma^2})^V$. In addition, this forcing does preserve the stationarity of S and does not create new d-approachable points of cofinality μ. All in all, S does not intersect $S(d)^{V[G*H]}$. However, since we collapsed $(\lambda^{+\gamma})^V$, the coloring d is no longer normal. Nevertheless, this is still enough to conclude $\aleph_{\gamma^2+1} \cap \mathrm{cof}(\mu) \notin I[\aleph_{\gamma^2+1}]$.

Claim 4 ([GS08, Lemma 2.20])**.** *There is a normal, subadditive coloring* $\overline{d}$ *on* $(\lambda^+)^{V[G*H]}$ *such that* $S(\overline{d})$ *does not contain a club relative to cofinality* μ.

[18] Technically, the proof of [Pov20, Lemma 3.18] is done for Sinapova's forcing without interleaved collapses, but a similar proof goes through even if one interleaves collapses.

[19] To show this, one just forces below a condition whose measure one sets concentrate on sets x whose associated Prikry points are above μ.

Proof. Let d' be any normal, subadditive coloring on $(\lambda^+)^{V[G*H]}$. Define $\overline{d}(\alpha,\beta) := \max(d(\alpha,\beta), d'(\alpha,\beta))$. It is easy to see that $\overline{d}$ is subadditive. Because $d(\alpha,\beta) \leq \overline{d}(\alpha,\beta)$ for all $\alpha < \beta < (\lambda^+)^{V[G*H]}$, any ordinal which is $\overline{d}$-approachable must also be d-approachable and so $S(\overline{d}) \subseteq S(d)$.

Lastly, the fact that $d'(\alpha,\beta) \leq \overline{d}(\alpha,\beta)$ for all $\alpha < \beta < (\lambda^+)^{V[G*H]}$ implies that $\overline{d}$ is normal: Let $\alpha < (\lambda^+)^{V[G*H]}$ and $i < \mathrm{cf}(\gamma)$. Then

$$|\{\beta < \alpha \mid \overline{d}(\beta,\alpha) \leq i\}| \leq |\{\beta < \alpha \mid d'(\alpha,\beta) \leq i\}| < \lambda$$

as desired. □

In $V[G*H]$, $\aleph_{\gamma^2}$ is a strong limit cardinal at which the SCH fails. □

4. Failure of Approachability at every Singular Cardinal

The main result of this section is Theorem B. Namely:

Theorem 4.1. *Suppose that there is a class of supercompact cardinals. Then, there exists a class forcing extension in which AP_κ fails for every singular cardinal κ.*

Moreover, for each singular cardinal κ, there are unboundedly many regular cardinals $\delta < \kappa$ for which $\kappa^+ \cap \mathrm{cof}(\delta) \notin I[\kappa^+]$.

From the above we immediately deduce the following corollary:

Corollary 4.2. *Suppose that there is a class of supercompact cardinals. Then, there exists a model of ZFC without special κ^+-Aronszajn trees any singular cardinal κ.*

The proof of this theorem consists of two parts: Given a regular cardinal μ and a supercompact cardinal $\kappa > \mu$, first we construct a forcing $\mathbb{A}(\mu,\kappa)$ that collapses κ to become μ^+ which when followed with a class product of Levy collapses $\mathbb{C}$, yields a two-step iteration $\mathbb{A}(\mu,\kappa) * \dot{\mathbb{C}}$ that forces $\delta^+ \cap \mathrm{cof}(\mu) \notin I[\delta^+]$ for every singular cardinal $\delta > \kappa$ with $\mathrm{cf}(\delta) < \mu$. In the second part we show how to iterate this construction to produce a model where AP_δ fails for every singular δ.

Let us begin defining the poset $\mathbb{A}(\mu,\kappa)$. Recall that if κ is a supercompact cardinal, $\ell\colon \kappa \to V_\kappa$ is called a *Laver function* if for each set x and a regular cardinal $\Theta \geq \kappa$ such that $x \in H(\Theta)$ there is an elementary embedding $j\colon V \to M$ with critical point κ, $j(\kappa) > \Theta$, M is closed under Θ-sequences of its members and $j(\ell)(\kappa) = x$. In [Lav78], Laver showed that Laver functions always exist. This makes sense of the next definition:

Definition 4.3 (The poset $\mathbb{A}(\kappa,\mu)$)**.** Let μ be regular and $\kappa > \mu$ supercompact. Fix a Laver function $\ell\colon \kappa \to V_\kappa$. We define a poset $\mathbb{A}(\mu,\kappa)$ as the limit of an Easton-support Magidor iteration of Prikry-type forcings $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha)_{\alpha<\kappa}$ with the following iterands: Suppose we have defined $(\mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta)_{\beta<\alpha}$. We let $\mathbb{P}_\alpha$ be the Easton-support Magidor limit of $(\mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta)_{\beta<\alpha}$. We let $\dot{\mathbb{Q}}_\alpha$ be defined as follows:

- **Case 1:** Suppose that α is inaccessible, $|\mathbb{P}_\beta| < \alpha$ for every $\beta < \alpha$ and

$$\ell(\alpha) = (\dot{\mathbb{L}}_\alpha, \langle \delta_i \mid i < \gamma\rangle, \dot{\vec{\mathcal{I}}}, \dot{\vec{\mathcal{B}}})$$

  where
  - $\dot{\mathbb{L}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a $<\alpha$-strategically closed poset.
  - $\Vdash_{\mathbb{P}_\alpha * \dot{\mathbb{L}}_\alpha}$ "$\mathrm{SLIP}(\check{\mu}, \langle \check{\delta}_i \mid i < \check{\gamma}\rangle)$ holds as witnessed by $\dot{\vec{\mathcal{I}}}$ and $\dot{\vec{\mathcal{B}}}$".

In this case, we let $\delta^*_\alpha := (\sup_{i<\gamma}\delta_i)^+$ and let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for

$$\dot{\mathbb{L}}_\alpha * \dot{\mathbb{P}}(\check{\mu}, \vec{\check{\delta}}, \vec{\dot{\mathcal{I}}}, \vec{\dot{\mathcal{B}}}) * \dot{\mathrm{Coll}}(\check{\mu}, \check{\delta}^*_\alpha).$$

- **Case 2:** Otherwise, we let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\dot{\mathrm{Coll}}(\check{\mu}, |\check{\mathbb{P}}_\alpha|)$.

We first justify the following easy properties of $\mathbb{A}(\mu,\kappa)$:

Lemma 4.4. *Let μ be regular and $\kappa > \mu$ supercompact.*

1. *$(\mathbb{A}(\mu,\kappa), \leq, \leq_0)$ is a Prikry-type forcing and in fact with pure μ-decidability.*
2. *$(\mathbb{A}(\mu,\kappa), \leq)$ and $(\mathbb{A}(\mu,\kappa), \leq_0)$ are κ-cc.*
3. *$(\mathbb{A}(\mu,\kappa), \leq_0)$ is $<\mu$-directed closed.*
4. *$(\mathbb{A}(\mu,\kappa), \leq)$ forces "$\mu^+ = \kappa$".*

Proof. Point (1) follows from Lemma 2.14, together with Lemma 3.24, noticing that we have assumed $\delta_0 > \mu$ and thus that the intermediate forcing has pure μ-decidability by virtue of Lemma 3.24. Both parts of point (2) follow easily using a Δ-system argument. Point (3) follows from Lemma 2.15. Point (4) follows from standard genericity considerations. □

The following result is instrumental for our main theorem. Essentially, it says that after forcing with $\mathbb{A}(\mu,\kappa)$ not only $\{\alpha < \delta^+ \mid \alpha \text{ is not } d\text{-approachable}\}$ is stationary, but also it remains stationary (and hence, non-approachable) in the generic extension by any $<\kappa$-directed closed forcing $\mathbb{P}$.

Theorem 4.5 (Indestructibility of $\neg\mathsf{AP}$)**.** *Let μ be regular and $\kappa > \mu$ supercompact. Let $\gamma < \mu$ be a limit ordinal and $\vec{\delta} = \langle \delta_i \mid i < \gamma\rangle$ an increasing and continuous sequence of regular cardinals with $\delta_0 \geq \kappa$. Let $\delta := \sup_{i<\gamma}\delta_i$ and $d\colon [\delta^+]^2 \to \mathrm{cf}(\gamma)$ be a normal, transitive coloring.*

Let G be $\mathbb{A}(\mu,\kappa)$-generic and work in $V[G]$. Whenever $\mathbb{P}$ is a $<\kappa$-directed closed poset which preserves δ^+, every δ_i and forces $\mathrm{SLIP}(\mu,\vec{\delta})$, for any large enough regular cardinal Θ there are stationarily many $M \in [H^{V[G]}(\Theta)]^{<\kappa}$ such that:

1. *$M \prec H^{V[G]}(\Theta)$.*
2. *$\mathbb{A}(\mu,\kappa), \mathbb{P}, d \in M$.*
3. *$\mathrm{cf}(\sup(M\cap\delta^+)) = \mu$ and $\sup(M\cap\delta^+)$ is not d-approachable.*
4. ***(Existence of master conditions)** Whenever $p \in \mathbb{P}\cap M$ there is a condition $p^* \in \mathbb{P}$ with $p^* \leq p$ such that whenever $D \in M$ is open dense in $\mathbb{P}$, there is $q \in D \cap M$ with $p^* \leq q$.*

Proof. Fix parameters as in the theorem and assume that they belong to $H(\Theta)$. In $V[G]$, let $F\colon [H^{V[G]}(\Theta)]^{<\omega} \to [H^{V[G]}(\Theta)]^{<\kappa}$ be any function. We want to find $M \in [H^{V[G]}(\Theta)]^{<\kappa}$ which is closed under F and satisfies the additional requirements (1)–(3). To this end, let $\dot{F}$ and $\dot{\mathbb{P}}$ be $\mathbb{A}(\mu,\kappa)$-names with $\dot{F}^G = \dot{F}$ and $\dot{\mathbb{P}}^G = \mathbb{P}$. Assume without loss of generality that the empty condition of $\mathbb{A}(\mu,\kappa)$ forces $\dot{\mathbb{P}}$ to have the claimed properties. Let $\dot{\mathcal{I}}_i, \dot{\mathcal{B}}_i$ be $\mathbb{P}$-names forced to witness $\mathrm{SLIP}(\check{\mu}, \vec{\check{\delta}})$ and $\Theta' > \Theta$ be a big enough regular cardinal such that $\dot{F}, \dot{\mathbb{P}}, \dot{\mathcal{I}}_i, \dot{\mathcal{B}}_i \in H^V(\Theta')$.

Let $j\colon V \to M$ be a $|H^V(\Theta')|$-supercompact embedding for κ such that

$$j(\ell)(\kappa) = (\dot{\mathbb{P}}, \langle \delta_i \mid i<\gamma\rangle, \vec{\dot{\mathcal{I}}}, \vec{\dot{\mathcal{B}}}).$$

It follows that in M, we have

$$j(\mathbb{A}(\mu,\kappa)) = \mathbb{A}(\mu,\kappa) * (\dot{\mathbb{P}} * \dot{\mathbb{P}}(\check{\mu}, \vec{\check{\delta}}, \vec{\dot{\mathcal{I}}}, \vec{\dot{\mathcal{B}}}) * \dot{\mathrm{Coll}}(\check{\mu}, \check{\delta}^+)) * j(\dot{\mathbb{A}})_{\kappa+1, j(\kappa)}$$

where $j(\dot{\mathbb{A}})_{\kappa+1,j(\kappa)}$ is forced to be an Easton support Magidor iteration of Prikry-type forcings with pure $\check{\mu}$-decidability and a $<\check{\mu}$-closed pure extension ordering.

Let H be $j(\mathbb{A}(\mu,\kappa))$-generic over V such that $j[G]\subseteq H$. Then j lifts to

$$j\colon V[G]\to M[H].$$

Let $N':=j[H^V(\Theta')][H]$ and $N:=N'\cap H^{M[H]}(j(\Theta))$. We want to show that in $M[H]$, N is as required by the theorem with j applied to the parameters.

Claim 1. *$j(\mathbb{A}(\mu,\kappa)), j(\mathbb{P}), j(d)\in N$.*

Proof of claim. This is clear by our choice of Θ'. □

Claim 2. *$N\prec H^{M[H]}(j(\Theta))$, $|N|<j(\kappa)$ and $j(F)$"$[N]^{<\omega}\subseteq N$.*

Proof of claim. Clearly, $|N|<j(\kappa)$ as $j(\kappa)>\Theta'=|N|$. It is also routine to check that $N'\prec H^{M[H]}(j(\Theta'))$. Using this one can show that N is closed under $j(F)$: Given any $(x_0,\dots,x_k)\in[N]^{<\omega}$ there are $(\tau_0,\dots,\tau_n)\in H^V(\Theta')$ such that $x_i=j(\tau_i)^H$. By definition, $j(\dot F)^H(j(\tau_0)^H,\dots,j(\tau_k)^H)=j(\dot F^G(\tau_0^G,\dots,\tau_k^G))\in j[H^{V[G]}(\Theta)]\subseteq H^{M[H]}(j(\Theta'))$. In addition, this very same value of $j(F)$ belongs to $N':=j[H^V(\Theta')][H]$ as this is an elementary submodel of $H^{M[H]}(j(\Theta'))$.

Finally, we argue that $N\prec H^{M[H]}(j(\Theta))$. Suppose that $H^{M[H]}(j(\Theta))\models$ "$\exists x\varphi(x,a)$" for some $a\in N$. Then, the following holds in the bigger model $H^{M[H]}(j(\Theta'))$:

$$\text{"There is } b\in H^{M[H]}(j(\Theta)) \text{ such that } H^{M[H]}(j(\Theta))\models\varphi(b,a)\text{"}.$$

This is a first-order formula with parameters $\{a,b,j(\Theta)\}$, all of which available to the model N'. Thus, by elementarity, there is $b\in N'\cap H^{M[H]}(j(\Theta))=N$ with the previous property. As a result $N\prec H^{M[H]}(j(\Theta))$. □

Claim 3. $\mathrm{cf}^{M[H]}(\sup(N\cap j(\delta^+)))=\mu$.

Proof of claim. Since $j(\kappa)>\Theta'$, we have $N'\cap j(\kappa)\in j(\kappa)$. Ergo, since H is generic for a $j(\kappa)$-cc forcing, it is a standard fact that $N'\cap M=j[H^V(\Theta')]$. In particular, $N\cap j(\delta^+)=N'\cap j(\delta^+)=j[\delta^+]$. Because of $\mathrm{Coll}(\mu,\delta^+)$, $\mathrm{cf}^{M[H]}(\sup j[\delta^+])=\mu$.

For the reader's benefit we provide the proof of the standard fact.

The $\supseteq$-inclusion is clear. For the $\subseteq$-inclusion, let $\tau^H\in N'\cap M$ with $\tau\in j[H^V(\Theta')]$. Since $j[H^V(\Theta')]\prec j(H^V(\Theta'))$, there is a maximal antichain $A\in j[H^V(\Theta')]$ of conditions forcing $\tau=\check{x}$ for some $x\in j(H^V(\Theta'))$. Since $j(\mathbb{Q})$ is $j(\kappa)$-cc, there is an enumeration $f\colon\mu\to A$ for some cardinal $\mu<j(\kappa)$. By elementarity,

$$\mu\in j[H^V(\Theta')]\cap j(\kappa)=N'\cap j(\kappa)=\kappa\subseteq j[H^V(\Theta')].$$

It follows that $A=f[\mu]\subseteq j[H^V(\Theta')]$. In particular, there is a condition $q\in H\cap j[H^V(\Theta')]$ forcing "$\tau=\check{x}$" for some $x\in j(H^V(\Theta'))$. By elementarity, $x\in j[H^V(\Theta')]$. Thus, $\tau^H=x\in j[H^V(\Theta')]$, as needed. □

Let us denote $\rho:=\sup(j[\delta^+])=\sup(N\cap j(\delta^+))$.

Claim 4. *ρ is not $j(d)$-approachable in $M[H]$.*

Proof of claim. In M, $\mathrm{cf}(\rho)=\delta^+$. δ^+ is regular after forcing with $\mathbb{A}(\mu,\kappa)*\dot{\mathbb{P}}$ and is collapsed to have size and cofinality μ after forcing with

$$\mathbb{A}(\mu,\kappa)*\dot{\mathbb{P}}*\dot{\mathbb{P}}(\check{\vec{\delta}},\dot{\vec{\mathcal{I}}},\dot{\vec{\mathcal{B}}})*\dot{\mathrm{Coll}}(\check{\mu},\check{\delta}^+).$$

This is still the case in $M[H]$ because μ is preserved. In summary,

$$M[H] \models \text{“}\operatorname{cf}(\rho) = \operatorname{cf}(\delta^+) = \mu\text{”}.$$

So we have to show that ρ is not $j(d)$-approachable in $M[H]$. As in the proof of Theorem 3.31, in M, there is a normal transitive coloring e on δ^+ such that in any set-sized forcing extension, ρ is not $j(d)$-approachable if and only if δ^+ is not e-approachable. After forcing with $\mathbb{A}(\mu,\kappa) * \dot{\mathbb{P}}$, by assumption, δ^+ and every δ_i are cardinals and therefore e is still a normal, transitive coloring. By Corollary 3.29, δ^+ is not e-approachable after forcing with

$$\dot{\mathbb{P}}(\check{\vec{\delta}}, \dot{\vec{\mathcal{I}}}, \dot{\vec{\mathcal{B}}}) * \dot{\operatorname{Coll}}(\check{\mu}, \check{\delta}^+).$$

Ergo, ρ is not $j(d)$-approachable in $M[H \restriction \kappa+1]$ where $H \restriction \kappa+1$ is the $\mathbb{A}(\mu,\kappa) * \dot{\mathbb{P}} * \dot{\mathbb{P}}(\check{\vec{\delta}}, \dot{\vec{\mathcal{I}}}, \dot{\vec{\mathcal{B}}}) * \dot{\operatorname{Coll}}(\check{\mu}, \check{\delta}^+)$-generic filter induced by H.

Let us show that this is preserved by the tail forcing

$$\mathbb{R} := j(\dot{\mathbb{A}}(\mu,\kappa))^{H\restriction\kappa+1}_{\kappa+1, j(\kappa)}.$$

As stated before, $\mathbb{R}$ is a Prikry-type poset with pure μ-decidability and a $<\mu$-closed pure extension ordering. In $M[H \restriction \kappa+1]$, let $\Delta\colon \mu \to \rho$ be increasing and cofinal. In $M[H]$, ρ is $j(d)$-approachable if and only if there is an unbounded subset of $\operatorname{range}(\Delta)$ on which $j(d)$ is bounded (by Fact 2.28). So assume that $\dot{\Gamma}$ is an $\mathbb{R}$-name for an increasing function from $\check{\mu}$ into $\operatorname{range}(\check{\Delta})$ such that $j(d)$ is bounded on its image (say, with value $\check{\zeta}$), forced by some $r \in \mathbb{R}$. We construct an increasing sequence $\langle \alpha_i \mid i<\mu\rangle$ of elements of μ and a $\leq_0$-decreasing sequence $\langle r_i \mid i<\mu\rangle$ of elements of $\mathbb{R}$ such that for any $i<\mu$, $r_i \Vdash \dot{\Gamma}(\check{i}) = \check{\Delta}(\check{\alpha}_i)$. Assume that $\langle \alpha_j \mid j<i\rangle$ and $\langle r_j \mid j<i\rangle$ have been constructed. Let r'_i be a $\leq_0$-lower bound of $\langle r_j \mid j<i\rangle$ (or $r_i = r$ if $i=0$). Then $r'_i \Vdash \exists \alpha < \check{\mu}(\dot{\Gamma}(\check{i})) = \check{\Delta}(\check{\alpha})$, so by the pure μ-decidability of $\mathbb{R}$ we can find α_i and $r_i \leq_0 r'_i$ such that

$$r_i \Vdash \dot{\Gamma}(\check{i}) = \check{\Delta}(\check{\alpha}_i)$$

Since $\dot{\Gamma}$ is forced to be increasing and Δ is increasing, $\alpha_i > \alpha_j$ for all $j<i$. Now we claim that $\{\Delta(\alpha_i) \mid i<\mu\}$ is a cofinal subset of ρ on which $j(d)$ is bounded with value ζ, obtaining a contradiction as that set is in $M[H \restriction \kappa+1]$. First observe that, because the sequence $\langle \alpha_i \mid i<\mu\rangle$ is increasing, the set $\{\alpha_i \mid i<\mu\}$ is cofinal in μ and so $\{\Delta(\alpha_i) \mid i<\mu\}$ is cofinal in ρ. Lastly, whenever $j<i<\mu$, we have

$$r_i \Vdash j(\check{d})(\check{\Delta}(\check{\alpha}_j), \check{\Delta}(\check{\alpha}_i)) = j(\check{d})(\dot{\Gamma}(\check{j}), \dot{\Gamma}(\check{i})) \leq \check{\zeta}$$

so actually $j(d)(\Delta(\alpha_j), \Delta(\alpha_i)) \leq \zeta$. □

We complete the argument showing that N admits “master conditions” for $j(\mathbb{P})$.

Claim 5. *Whenever $p \in j(\mathbb{P}) \cap N$ there is a condition $p^* \in j(\mathbb{P})$ with $p^* \leq p$ such that whenever $D \in N$ is dense open there is $q \in D \cap N$ such that $p^* \leq q$.*

Proof of claim. Fix $j(p) \in j(\mathbb{P}) \cap N$. By construction, we know that in $M[H]$ there is a filter K which is $\mathbb{P}$-generic over $M[G]$, which we can assume has p as a member. By our assumptions, we know that M is closed under $|\mathbb{P}|$-sequences and so $j \restriction \mathbb{P}$ is a member of $M[G]$. In particular, $j[K] := (j \restriction \mathbb{P})[K]$ is also a member of $M[H]$, since $K \in M[H]$. We have $|K| < \Theta' < j(\kappa)$ and so by the $<j(\kappa)$-directed closure of $j(\mathbb{P})$ in $M[H]$ there is a condition $p^* \in j(\mathbb{P})$ which is a lower bound of $j[K]$. We claim that p^* is required. To this end, let $D \in N$ be open dense in $j(\mathbb{P})$. By the

definition of N, $D = j(E)$ for $E \in H^{V[G]}(\Theta')$, E open dense in $\mathbb{P}$. It follows that there is $q \in E \cap K$. Then $j(q) \in D \cap j[K] \cap N$ and by construction, $p^* \leq j(q)$. □

This finishes the proof. □

We obtain the following generalization to our answer of Shelah's problem:

Theorem 4.6. *Assume the* GCH *and that there is a supercompact cardinal κ together with a proper class of measurables. Let $\mu < \kappa$ be regular. Then, there is a model of* ZFC *in which $\delta^+ \cap \mathrm{cof}(\mu) \notin I[\delta^+]$ for every singular $\delta > \kappa$ with $\mathrm{cf}(\delta) < \mu$.*

Proof. Let G be $\mathbb{A}(\mu, \kappa)$-generic. Standard arguments show that the GCH pattern survives in $V[G]$. In $V[G]$, let $\langle \kappa_i \mid i \in \mathrm{On} \rangle$ be an increasing and continuous sequence such that $\kappa_0 = \kappa$ and κ_i is measurable whenever i is a successor. (Note that we allow the existence of measurable cardinals that are limit of measurables.)

We force with $\mathbb{C}$, the On-length Easton support product of $\mathrm{Coll}(\kappa_i^{++}, {<}\kappa_{i+1})$. By the increasing closure of the posets (and Easton's lemma, Fact 2.10) one can show that $\mathbb{C}$ is *tame* in the sense of Friedman [Fri11]. In particular, all the standard results from forcing theory apply to $\mathbb{C}$ and the resulting generic extension is a model of ZFC. In the extension in $\mathbb{C}$, all cardinals below $\kappa = \mu^+$ survive. Standard arguments show that above κ all the V-cardinals of the form κ_i, κ_i^+ and κ_i^{++} remain cardinals while the rest are collapsed. It also follows that the only singular cardinals above κ in the extension are the κ_i's that were formerly singular (see Lemma 4.11).

Let $C \subseteq \mathbb{C}$ be $V[G]$-generic. For each $i \in \mathrm{On}$ denote

$$\mathbb{C}^i := \prod_{j \geq i} \mathrm{Coll}(\kappa_j^{++}, {<}\kappa_{j+1}) \text{ and } \mathbb{C}_i := \prod_{j < i} \mathrm{Coll}(\kappa_j^{++}, {<}\kappa_{j+1}).$$

Working in $V[G * C]$, let $\delta > \kappa$ be a singular cardinal with $\mathrm{cf}^{V[C]}(\delta) < \mu$. Then, there exists a limit ordinal γ such that $\delta = \kappa_\gamma$.

Note that $\delta^+ \cap \mathrm{cof}(\mu) \notin I[\delta^+]$ holds in $V[G * C]$ if and only if $\delta^+ \cap \mathrm{cof}(\mu) \notin I[\delta^+]$ holds in $V[C \cap \mathbb{C}_i]$ because the tail forcing $\mathbb{C}^\gamma$ is κ_γ^{++}-distributive in $V[C \cap \mathbb{C}_\gamma]$ (by Easton's lemma). Thus, it will be sufficient to check this latter property.

For each $i < \gamma$, κ_i is a measurable cardinal in $V[G]$ and so $\mathrm{Coll}(\kappa_i^{++}, {<}\kappa_{i+1})$ is a $\mathsf{LIP}(\kappa_i^{++}, \kappa_i)$-forcing. Combining this with Lemma 3.9 we infer that $\mathbb{C}_\gamma$ forces $\mathrm{SLIP}(\kappa_0, \langle \kappa_i \mid i < \gamma \rangle)$. So $\mathbb{C}_\gamma$ satisfies the requirements of Theorem 4.5. Let $d \colon [\kappa_i^+]^2 \to \mathrm{cf}(\gamma)$ be a normal, transitive coloring in V and denote $S := \delta^+ \cap \mathrm{cof}(\mu)$ as computed in $V[G]$. Note that the definition of this set is absolute between $V[G]$ and $V[G * C]$. By Fact 2.29, it suffices to show that S remains stationary after forcing with $\mathbb{C}_\gamma$ and that it consists of points that are not d-approachable.

Claim 6. *Working over $V[G]$, $\mathbb{C}_\gamma$ forces that $\check{S}$ is stationary and $\check{S} \cap \dot{S}(\check{d}) = \check{\emptyset}$.*

Proof of claim. Work in $V[G]$. We perform a density argument. Suppose that $\dot{C}$ is a $\mathbb{C}_\gamma$-name for a club in δ^+ as forced by a condition $p \in \mathbb{C}_\gamma$. By Theorem 4.5, there is $M \prec H^{V[G]}(\Theta)$ and $p \in \mathbb{C}_\gamma$ such that $\rho := \sup(M \cap \delta^+)$ non d-approachable, $\mathrm{cf}(\rho) = \mu$, and p^* is a $(\mathbb{C}_\gamma, M)$-master condition with $p^* \leq p$. Recall that this means that whenever $D \in M$ is open dense in $\mathbb{C}_\gamma$, there is $q \in D \cap M$ with $p^* \leq q$.

We claim that p^* forces that $\check{\rho} \in \dot{C} \cap \check{S}$. To this end, it suffices to show that p^* forces $\dot{C} \cap \check{\rho}$ to be unbounded in $\check{\rho}$. Given any $\zeta < \rho$, let $\zeta' > \zeta$ be with $\zeta' \in M$. The open dense set of all conditions in $\mathbb{C}_\gamma$ forcing $\check{\eta} \in \dot{C}$ for some $\eta > \zeta'$ is in M. As a result, there is a condition $q \in D \cap M$ such that $p^* \leq q$. By elementarity, there is a witnessing η in M. Therefore, p^* forces $\dot{C}$ to be unbounded in $\check{\rho}$.

Lastly, p^* also forces "ρ is not d-approachable" because $\mathbb{C}_\gamma$ adds no new subsets of μ and $\mathrm{cf}(\rho) = \mu$. (So, in fact, the weakest condition forces this property).

By density it follows that $\Vdash_{\mathbb{C}_\gamma}$ "$\check{S}$ is stationary and $\check{S} \cap \dot{S}(\check{d}) = \check{\emptyset}$". □

We are done with the theorem. □

Now we iterate the construction. To obtain enough instances of LIP along the way we alternate the forcing $\mathbb{A}$ with the Levy-collapse. Additionally, in order to show that there is a forcing which projects onto our tail forcing and forces appropriate instances of the SLIP-property, we leave a gap of two cardinals between the collapses (so that Lemma 3.9 applies). This necessarily leads to gaps between the cofinalities at which AP fails. We do not know if this is necessary, see Question 5.1.

Definition 4.7. Assume GCH. Let $\langle \kappa_\alpha \mid \alpha \in \mathrm{On} \rangle$ be an increasing and continuous sequence of cardinals such that $\kappa_0 = \aleph_0$ and κ_α is supercompact whenever α is a successor ordinal. We define an Easton support Magidor iteration

$$((\mathbb{P}_\alpha, \leq_\alpha, \leq_{\alpha,0}), (\dot{\mathbb{Q}}_\alpha, \dot{\leq}_\alpha, \dot{\leq}_{\alpha,0}))_{\alpha \in \mathrm{On}}$$

as follows: Assume that $(\mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta)_{\beta<\alpha}$ has been defined. Let $\mathbb{P}_\alpha$ be the Easton limit.

- **Case 1:** α is even. Then we let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\dot{\mathrm{Coll}}(\kappa_\alpha^{++}, <\kappa_{\alpha+1})$.
- **Case 2:** α is odd. Then we let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\dot{\mathbb{A}}(\kappa_\alpha^{++}, \kappa_{\alpha+1})$.

Definition 4.8. Let $\beta < \alpha$ be ordinals. We let $(\dot{\mathbb{P}}_{\beta,\alpha}, \dot{\leq}_{\beta,\alpha}, \dot{\leq}_{(\beta,\alpha),0})$ be a $\mathbb{P}_\beta$-name for the poset witnessing that $\mathbb{P}_\beta * \dot{\mathbb{P}}_{\beta,\alpha} \cong \mathbb{P}_\alpha$.

Remark 4.9. $1_{\mathbb{P}_\beta} \Vdash$ "$\mathbb{P}_{\beta,\alpha}$ is the Easton support Magidor iteration of $(\dot{\mathbb{P}}_{\beta,\gamma}, \dot{\mathbb{Q}}_\gamma)_{\gamma \in [\beta,\alpha)}$".

Lemma 4.10. *Let $\beta < \alpha$ be ordinals. $\mathbb{P}_\beta$ forces that $\dot{\mathbb{P}}_{\beta,\alpha}$ is a Prikry-type poset with pure κ_β^{++}-decidability and with a $<\kappa_\beta^{++}$-closed pure extension ordering.*

Additionally, if α is a limit, $\dot{\mathbb{P}}_{\beta,\alpha}$ is forced to have almost pure κ_α^+-decidability.

Proof. The decidability of $\dot{\mathbb{P}}_{\beta,\alpha}$ follows from Lemma 2.14 together with Lemma 4.4. Lemma 2.14 also implies the almost pure κ_α^+-decidability, as every $\dot{\mathbb{P}}_{\beta,\gamma}$ has it for $\gamma < \alpha$ since it is of size $< \kappa_\alpha^+$. The closure of the pure extension ordering follows from Lemma 2.15. □

So in particular, after forcing with $\mathbb{P}_\beta$, the tail forcing adds no new subsets of κ_β. This implies that we can define the class forcing extension as the "limit" of the forcing extensions by $\mathbb{P}_\beta$ for $\beta \in \mathrm{On}$. Abusing notation a bit, we will refer to this class forcing extension as an extension "by $\mathbb{P}$", treating $\mathbb{P}$ as the Easton support limit of the class iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha)_{\alpha \in \mathrm{On}}$.

Lemma 4.11. *After forcing with $\mathbb{P}$, the class of cardinals consists of $\aleph_0$ as well as*

$$\{\kappa_\alpha^{+i} \mid \alpha \in \mathrm{On},\ i \in \{0,1,2\}\}.$$

Proof. It is clear that all V-cardinals which are not as in the statement of the lemma are collapsed. We show that all the claimed cardinals are preserved. To this end, let $\alpha \in \mathrm{On}$. Since $\dot{\mathbb{P}}_{\alpha,\beta}$ is forced to be a Prikry-type forcing with a $<\kappa_\alpha^{++}$-closed pure extension ordering and pure κ_α^{++}-decidability, it suffices to show that $\mathbb{P}_\alpha$ preserves

$$\kappa_\alpha,\ \kappa_\alpha^+ \text{ and } \kappa_\alpha^{++}.$$

First assume that α is a successor. It follows that $\mathbb{P}_{\alpha-1}$ has size $< \kappa_\alpha$ and thus preserves κ_α. Let $G_{\alpha-1}$ be $\mathbb{P}_{\alpha-1}$-generic. Then $\dot{\mathbb{Q}}_{\alpha-1}^{G_{\alpha-1}}$ is κ_α-cc (by standard facts

for the Levy collapse or by Lemma 4.4 for $\mathbb{A}$) and thus preserves κ_α, κ_α^+ and κ_α^{++}. Since $\mathbb{P}_{\alpha-1} * \dot{\mathbb{Q}}_{\alpha-1} = \mathbb{P}_\alpha$, we are done.

Now assume α is a limit. By the previous paragraph it follows that a cofinal set of cardinals below κ_α is preserved by $\mathbb{P}$ and thus κ_α is preserved as well. Now assume in the first case that κ_α is singular. It follows that if κ_α^+ is collapsed by $\mathbb{P}_\alpha$, that poset forces $|\kappa_\alpha^+| = \kappa_\alpha$ and thus $\mathrm{cf}(\kappa_\alpha^+) < \kappa_\alpha$. However, this situation is not possible, since for every $\beta < \alpha$, $\mathbb{P}_\alpha \cong \mathbb{P}_\beta * \dot{\mathbb{P}}_{\beta,\alpha}$ where $|\mathbb{P}_\beta| < \kappa_\alpha$ (and thus cannot collapse κ_α^+) and $\dot{\mathbb{P}}_{\beta,\alpha}$ is a Prikry-type forcing with almost pure κ_α^+-decidability and a $< \kappa_\beta^+$-closed pure extension odering (thus it cannot add cofinal subsets of κ_α^+ of ordertype $\leq \kappa_\beta$). For κ_α^{++}, simply note that because of the GCH, $|\mathbb{P}_\alpha| \leq \kappa_\alpha^+$.

Lastly, assume in the second case that κ_α is regular. Then κ_α is in fact inaccessible and $|\mathbb{P}_\beta| < \kappa$ for every $\beta < \kappa$. We shall prove that $\mathbb{P}_\alpha$ preserves $\kappa_\alpha, \kappa_\alpha^+$ and κ_α^{++}. For κ_α^+ and κ_α^{++}, notice that $\mathbb{P}_\alpha$ has size κ_α and thus the κ_α^+-cc. For κ_α, assume that $\mathbb{P}_\alpha$ adds an unbounded subset of κ_α of ordertype $\gamma < \kappa_\alpha$. Let $\beta < \alpha$ be such that $\gamma < \kappa_\beta$. Since $\mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{\mathbb{P}}_{\beta,\alpha}$ and $\dot{\mathbb{P}}_{\beta,\alpha}$ is forced to have pure $< \kappa_\alpha$-decidability (as in Lemma 4.10), κ_α has cofinality $\leq \gamma$ after forcing with $\mathbb{P}_\beta$. But this is a contradiction, as $|\mathbb{P}_\beta| < \kappa_\alpha$. □

Now we turn toward showing that, in $V[\mathbb{P}]$, AP_δ fails for every singular cardinal δ. We intend to use Theorem 4.5. To apply this theorem, we need to find a poset which projects onto the direct extension ordering on a suitable tail forcing and forces appropriate instances of the SLIP-property.

Definition 4.12. Let $\beta < \alpha$ be ordinals. Let G_β be $\mathbb{P}_\beta$-generic and work in $V[G_\beta]$. We define $\mathbb{T}_{\beta,\alpha}$ as the Easton support product

$$\mathbb{T}_{\beta,\alpha} := \prod_{\gamma \in [\beta,\alpha)} \mathbb{T}((\dot{\mathbb{P}}_{\beta,\gamma}^{G_\beta}, \dot{\leq}_{\beta,\gamma}^{G_\beta}), (\dot{\mathbb{Q}}_\gamma^{G_\beta}, \dot{\leq}_{\gamma,0}^{G_\beta}))$$

where we are regarding $\dot{\mathbb{Q}}_\gamma$ as a $\mathbb{P}_\beta * \dot{\mathbb{P}}_{\beta,\gamma}$-name.

By Lemma 2.16 and standard arguments, one sees the following:

Lemma 4.13. *Let $\beta < \alpha$ be ordinals. Let G_β be $\mathbb{P}_\beta$-generic and work in $V[G_\beta]$. The poset $\mathbb{T}_{\beta,\alpha}$ is $< \kappa_\beta^{++}$-directed closed. Moreover, the identity defines a projection from $\mathbb{T}_{\beta,\alpha}$ onto $(\dot{\mathbb{P}}_{\beta,\alpha}^{G_\beta}, \dot{\leq}_{(\beta,\alpha),0}^{G_\beta})$.* □

We show that we can actually apply Theorem 4.5 to $\mathbb{T}_{\beta,\alpha}$:

Lemma 4.14. *Let $\beta < \alpha$ be ordinals such that β is even and α is a limit. Let $\langle \delta_i \mid i < \gamma \rangle$ for some limit ordinal γ enumerate the set $\{\kappa_j \mid j \in [\beta, \alpha) \text{ is odd}\}$. Then, inside $V[G_\beta]$, $\mathbb{T}_{\beta,\alpha}$ forces $\mathrm{SLIP}(\kappa_\beta^{++}, \langle \delta_i \mid i < \gamma \rangle)$.*

Proof. We intend to use Lemma 3.9 on the preservation of SLIP.

For each $i < \gamma$, let $\delta_i = \kappa_{j_i}$ for an odd $j_i \in (\beta, \alpha)$. Note that $\kappa_{j_{i-1}} = \kappa_{j_i - 2}$ for successor j_i. If $j_i = \beta + 1$ or $j_i - 1$ is a limit, let

$$\mathbb{P}_i := \mathbb{T}((\mathbb{P}_{\beta,j_i-1}, \leq_{(\beta,j_i-1)}), (\dot{\mathbb{Q}}_{j_i-1}^{G_\beta}, \dot{\leq}_{j_i-1,0}^{G_\beta}));$$

otherwise, $j_i - 1$ is a successor distinct from $\beta + 1$ in which case we let

$$\mathbb{P}_i := \mathbb{T}((\mathbb{P}_{\beta,j_i-1}, \leq_{(\beta,j_i-1)}), (\dot{\mathbb{Q}}_{j_i-1}^{G_\beta}, \dot{\leq}_{j_i-1,0}^{G_\beta})) \times \mathbb{T}((\mathbb{P}_{\beta,j_i}, \leq_{(\beta,j_i)}), (\dot{\mathbb{Q}}_{j_i}^{G_\beta}, \dot{\leq}_{j_i,0}^{G_\beta})).$$

It follows that $\mathbb{T}_{\beta,\alpha} = \prod_{i<\gamma} \mathbb{P}_i$.

Claim 1. *For each $i<\gamma$, $\mathbb{P}_i$ is an $\mathsf{LIP}((2^{\sup_{j<i}\delta_j})^+,\delta_i)$-forcing where*

$$(2^{\sup_{j<i}\delta_j})^+=\begin{cases}\kappa^{++}_{j_i-1}, & \text{if } j_i-1 \text{ is a limit or } j_i=\beta+1;\ [20]\\ \kappa^{++}_{j_i-2}, & \text{if } j_i-1 \text{ is a successor.}\end{cases}$$

Proof. Let $i<\gamma$. First, we compute $(2^{\sup_{j<i}\delta_j})^+$: If j_i-1 is a limit, then

$$(2^{\sup_{j<i}\delta_j})^+=\kappa^{++}_{j_i-1}$$

since $\sup_{j<i}\delta_j=\kappa_{j_i-1}$ and the GCH holds. If j_i-1 is a successor, then

$$\sup_{j<i}\delta_j=\delta_{i-1}=\kappa_{j_{i-1}}=\kappa_{j_i-2}$$

and so $(2^{\sup_{j<i}\delta_j})^+=\kappa^{++}_{j_i-2}$ by the GCH.

We have to discuss various cases:

Case $j_i=\beta+1$: Then, $\mathbb{P}_i$ is equivalent to the Levy-collapse

$$\mathrm{Coll}(\kappa^{++}_{j_i-1},{<}\kappa_{j_i})$$

which certainly is a $\mathsf{LIP}(\kappa^{++}_{j_i-1},\kappa_{j_i})$-forcing.

Case j_i-1 is a limit: Then $\mathbb{P}_i=\mathbb{T}((\mathbb{P}_{\beta,j_i-1},\leq_{\beta,j_i-1}),(\dot{\mathbb{Q}}^{G_\beta}_{j_i-1},\dot{\leq}^{G_\beta}_{j_i-1}))$.

Note that $|\mathbb{P}_{j_i-1}|\leq\kappa^+_{j_i-1}$ (because of the GCH) and $\dot{\mathbb{Q}}^{G_\beta}_{j_i-1}$ is a $\mathbb{P}_{j_i-1}$-name for the Levy collapse $\dot{\mathrm{Coll}}(\kappa^{++}_{j_i-1},{<}\kappa_{j_i})$ and this is forced to be a $\mathsf{LIP}(\kappa^{++}_{j_i-1},\kappa_{j_i})$-forcing. Thus, by Lemma 3.6 we infer that $\mathbb{P}_i$ is a $\mathsf{LIP}(\kappa^{++}_{j_i-1},\kappa_{j_i})$-forcing.

Case j_i-1 is a successor ordinal above β. Then we stipulated

$$\mathbb{P}_i=\mathbb{T}((\mathbb{P}_{\beta,j_i-2},\leq_{(\beta,j_i-2)}),(\dot{\mathbb{Q}}^{G_\beta}_{j_i-2},\dot{\leq}^{G_\beta}_{j_i-2,0}))\times\mathbb{T}((\mathbb{P}_{\beta,j_i-1},\leq_{(\beta,j_i-1)}),(\dot{\mathbb{Q}}^{G_\beta}_{j_i-1},\dot{\leq}^{G_\beta}_{j_i-1,0})).$$

By the previous argument the second factor is a $\mathsf{LIP}(\kappa^{++}_{j_i-1},\kappa_{j_i})$-forcing.

Regarding the first factor of the product; namely,

$$\mathbb{T}((\mathbb{P}_{\beta,j_i-2},\leq_{(\beta,j_i-2)}),(\dot{\mathbb{Q}}^{G_\beta}_{j_i-2},\dot{\leq}^{G_\beta}_{j_i-2,0}))$$

we have: $\dot{\mathbb{Q}}^{G_\beta}_{j_i-2}$ is a $\mathbb{P}_{j_i-2}$-name for $\dot{\mathbb{A}}(\kappa^{++}_{j_i-2},\kappa_{j_i-1})$, hence this first factor has size $<\kappa_{j_i}$ and is $<\kappa^{++}_{j_i-2}$-directed closed. By Lemma 3.5 we infer that $\mathbb{P}_i$ is a $\mathsf{LIP}(\kappa^{++}_{j_i-2},\kappa_{j_i})$-forcing, as sought. □

In summary, by Lemma 3.9, $\mathbb{T}_{\beta,\alpha}$ forces $\mathrm{SLIP}(\kappa^{++}_\beta,\langle\delta_i\mid i<\gamma\rangle)$. □

So now we can finally show our main theorem:

Proof of Theorem 4.1. It follows from our analysis of the cardinal structure that the class of singular cardinals in $V[\mathbb{P}]$ is given by the class of κ_α where α is a limit ordinal and $\kappa_\alpha>\alpha$. So let α be with that property. Fix in V a normal transitive coloring $d\colon[\kappa^+_\alpha]^2\to\mathrm{cf}(\kappa_\alpha)$. We want to show that in $V[\mathbb{P}]$, there are stationarily many points in κ^+_α which are not d-approachable (by Fact 2.29 this implies $\kappa^+_\alpha\notin I[\kappa^+_\alpha]$). To this end, it suffices to show that this is the case in $V[\mathbb{P}_\alpha]$, because for any $\gamma>\alpha$, $\dot{\mathbb{P}}_{\alpha,\gamma}$ is forced not to add any new subsets of κ^+_α by Lemma 4.10. So let $\beta<\alpha$ be an even successor ordinal such that $\alpha<\kappa_{\beta-1}$. Hence,

$$\mathbb{P}_\beta\cong\mathbb{P}_{\beta-1}*\dot{\mathbb{A}}(\kappa^{++}_{\beta-1},\kappa_\beta).$$

[20] As usual, the empty supremum is regarded as κ_β.

Let G_β be $\mathbb{P}_\beta$-generic and $\dot{S}$ be a $\mathbb{P}_{\beta,\alpha}^{G_\beta}$-name for the set of $\xi < \kappa_\alpha^+$ of $V[G_\beta]$-cofinality $\kappa_{\beta-1}^{++}$ that are not d-approachable in $V[G_\beta][\mathbb{P}_{\beta,\alpha}^{G_\beta}]$. As in Theorem 4.6, we claim:

Claim 2. *Working in $V[G_\beta]$, $\mathbb{P}_{\beta,\alpha}^{G_\beta}$ forces "$\dot{S}$ is stationary".*

Proof of clain. We perform a density argument. Let $p_0 \in \mathbb{P}_{\beta,\alpha}^{G_\beta}$ be arbitrary.

Let γ be a limit ordinal and let $\langle \delta_i \mid i < \gamma \rangle$ enumerate $\{\kappa_j \mid j \in [\beta, \alpha) \text{ is odd}\}$.[21] Let $\dot{C}$ be a $\dot{\mathbb{P}}_{\beta,\alpha}^{G_\beta}$-name for a club subset of κ_α^+ which, without loss of generality, is forced to be such by p_0. By Lemma 2.16, there is a projection

$$\pi \colon \mathbb{T}_{\beta,\alpha} \to (\dot{\mathbb{P}}_{\beta,\alpha}^{G_\beta}, \dot{\leq}_{(\beta,\alpha),0}^{G_\beta}).$$

Let $t_0 \in \mathbb{T}_{\beta,\alpha}$ such that $\pi(t_0) \leq p_0$. (This exists in that π is a projection.)

Now we apply Theorem 4.5 inside $V[G_\beta]$ to find (for Θ large enough regular) $M \prec H(\Theta)$ with $|M| < \kappa_\beta$ such that the following hold:

(1) M contains every relevant parameter (in particular, $\pi, \dot{C}, \dot{\mathbb{P}}_{\beta,\alpha}^{G_\beta}$, t_0 and $\mathbb{T}_{\beta,\alpha}$)
(2) $\operatorname{cf}(\sup(M \cap \kappa_\alpha^+)) = \kappa_{\beta-1}^{++}$ and $\sup(M \cap \kappa_\alpha^+)$ is not d-approachable.
(3) **(Master condition)** There is a condition $t \in \mathbb{T}_{\beta,\alpha}$, $t \leq t_0$, such that whenever $D \in M$ is open dense in $\mathbb{T}_{\beta,\alpha}$, there is $q \in D \cap M$ with $t \leq q$.

Note that we can indeed invoke Theorem 4.5 because by Lemma 4.13 and Lemma 4.14, $\mathbb{T}_{\beta,\alpha}$ is $<\kappa_\beta^{++}$-directed closed and forces $\mathrm{SLIP}(\kappa_\beta^{++}, \langle \delta_i \mid i < \gamma \rangle)$.

Let $p := \pi(t)$. Note that $p \leq p_0$ because π is order-preserving. We claim that p forces (in $(\dot{\mathbb{P}}_{\beta,\alpha}^{G_\beta}, \dot{\leq}_{\beta,\alpha}^{G_\beta})$) that $\sup(M \cap \kappa_\alpha^+)$ is in $\dot{C}$. We will show this by showing that p forces $\dot{C} \cap \sup(M \cap \kappa_\alpha^+)$ to be unbounded in $\sup(M \cap \kappa_\alpha^+)$. To this end, let $\zeta < \sup(M \cap \kappa_\alpha^+)$. We can assume $\zeta \in M \cap \kappa_\alpha^+$. Given any $q \in \dot{\mathbb{P}}_{\beta,\alpha}^{G_\beta}$, q forces that there exists some element of $\dot{C}$ above ζ. By the almost pure κ_α^+-decidability of $\dot{\mathbb{P}}_{\beta,\alpha}^{G_\beta}$ (see Lemma 4.10), there exists $r \dot{\leq}_{(\beta,\alpha),0}^{G_\beta} q$ and $\eta < \kappa_\alpha^+$ such that r forces that there exists some element of $\dot{C}$ in the interval (ζ, η). Ergo, the set D of all $r \in \dot{\mathbb{P}}_{\beta,\alpha}^{G_\beta}$ such that for some $\eta_r < \kappa_\alpha^+$, r forces $\dot{C} \cap (\zeta, \eta_r)$ to be nonempty, is open dense in $(\dot{\mathbb{P}}_{\beta,\alpha}^{G_\beta}, \dot{\leq}_{(\beta,\alpha),0}^{G_\beta})$. Since π is a projection, $\pi^{-1}[D]$ is open dense in $\mathbb{T}_{\beta,\alpha}$. Ergo, by Clause (3) above (i.e., t is a master condition) there exists $r \in \pi^{-1}[D] \cap M$ with $t \leq r$ and thus $\pi(r) \in D \cap M$ with $\pi(t) \leq \pi(r)$. The corresponding $\eta_{\pi(r)}$ is in M as well by elementarity. Ergo $\pi(t)$ forces that there is some element of $\dot{C}$ above ζ and below $\sup(M \cap \kappa_\alpha^+)$. Since ζ was arbitrary, $\pi(t)$ forces that $\dot{C}$ is unbounded in $\sup(M \cap \kappa_\alpha^+)$. In addition, the closure of $\dot{\mathbb{P}}_{\beta,\alpha}^{G_\beta}$ in the pure extension ordering implies that $\sup(M \cap \kappa_\alpha^+)$ is not d-approachable after forcing with that poset. Ergo, p forces "$\sup(M \cap \kappa_\alpha^+)$ is not d-approachable". Thus, p forces "$\dot{S} \cap \dot{C} \neq \emptyset$", as needed. □

We are done with the proof of the theorem. □

Remark 4.15. It follows easily from our proof that, after forcing with $\mathbb{P}$, for every singular cardinal κ_γ and every even $\beta < \gamma$ with $\alpha < \kappa_{\beta-1}$, the set $\kappa_\gamma^+ \cap \operatorname{cof}(\kappa_{\beta-1}^{++})$ is not a member of $I[\kappa_\gamma^+]$. In particular, there are stationarily many non-approachable points of arbitrarily large cofinality. This disposes with the moreover assertion in Theorem B.

[21] In particular, $\kappa_\alpha = \sup_{i<\gamma} \delta_i$

5. Discussion and Open Questions

There is an interesting contrast between the results of this paper and [Jak25]. Namely, the poset which we devised in this paper to collapse the successor of a singular cardinal without making it approachable has much stronger requirements than the corresponding one from [Jak25]. Due to this, we had to separate our collapses and thus were not able to obtain the failure of the approachability property for all possible cofinalities below the singular cardinal, simultaneously. Thus we ask:

Question 5.1. Suppose that λ is a singular cardinal of uncountable cofinality. Is there a model of ZFC plus "For every regular $\mu \in (\mathrm{cf}(\lambda), \lambda)$ there is a stationary subset of $\lambda^+ \cap \mathrm{cof}(\mu)$ which is not in $I[\lambda^+]$"?

Another relevant question regards the failure of the SCH. Answering a question from [BNLHU20], in [Git21b] Gitik produces a model of ZFC where both AP and SCH fail on a club class. A natural question is if this configuration can be combined with Theorem B. More precisely we ask:

Question 5.2. Assuming a proper class of supercompact cardinals, is it consistent with ZFC that the SCH fails on a club class and AP fails at every singular cardinal?

Combining Theorem B with a **global** failure of the SCH is a major open problem for it would yield a negative solution to the following long-standing question:

Question 5.3 (Woodin, 80's)**.** Is the configuration "$\aleph_\omega$ is strong limit and both SCH and AP fail at $\aleph_\omega$" consistent with ZFC?

By a result of Jensen in the model of Theorem B there are no special κ^+-Aronsjzan trees for no singular cardinal κ. Thus, it is natural to ask:

Question 5.4. Assuming appropriate large cardinals, is it possible to have a model of ZFC where the tree property holds at the successor of every singular cardinal?

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